Infinite-Valued Logic Based on Two-Valued Logic and Probability

Part 1.4. The TEE Model for Grades of Membership

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Abstract: This paper precisates the meaning of numerical membership values and shows that there is no contradiction between a probabilistic interpretation of grades of membership on the one hand, and membership functions of the attribute universe whose ordinates add up to more than 1 on the other. The membership value in a class λ, e.g., λ=tall, assigned by a subject to an object of a given attribute value u^{ex} (e.g., u^{ex}=exact height value) is interpreted as the subject's estimate of $P(\lambda|u^{ex})$, the probability that this object would be assigned (by herself or another subject) the label λ in the presence of fuzziness $\#1, 2$ or 3 (in an experimental or natural language LB (labeling) or YN (yes-no) situation in which the subject uses a *nonfuzzy* threshold criterion in the universe U of *estimated* attribute values). $\lambda = \lambda_l$ is assumed to be an element of a label set Λ, such as $\Lambda = \{\text{small, medium, tall}\}$. The probabilistic 'summing up to 1 requirement' applies to the sum of $P(\lambda_l|u^{ex}) = \mu_{\lambda_l}(u^{ex})$ over the elements λ_l of Λ. In 'traditional' fuzzy set theory, this requirement is expressed by the formula for the negation, $\mu_{NOT\,\lambda}(u^{ex}) + \mu_\lambda(u^{ex}) = 1 \quad \forall u^{ex}$, as well as by the 'summing up to 1' requirement (of the grades of membership of a given point u^{ex} in all clusters) used by fuzzy clustering algorithms. The shapes of the $P(\lambda_l|u^{ex}) = \mu_{\lambda_l}(u^{ex})$ membership curves are *derived* in the TEE model, and are contrasted with the shapes of the $P(u^{ex}|\lambda_l)$ probability curves for which the 'summing up to 1 over u^{ex}' holds. The significance of the membership values 0, 0.5 and 1, as well as the meaning of a 'subnormal fuzzy set', of the probability of a fuzzy event and of the possibility/probability consistency factor are precisated. Zadeh's *postulated* formulas for the last two quantities are *derived* and confirmed. Entropy expressions connected with fuzzy subsets are derived. The complementation paradox of fuzzy set theory is shown to disappear when the *postulated* max operator for OR is replaced by the operators *derived* from the TEE model.

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Figures and Contents of Infinite-Valued Logic Based on Two-Valued Logic and Probability Part 1.4. The TEE Model for Grades of Membership

The relation of the infinite-valued system to the calculus of probabilities awaits further inquiry

Karl Jan Lukasiewicz (Borkowsky, 1970 p. 173).

1. Overview

1.1 Introduction

In three previous papers (Hisdal 1986a,b, 1988a) we paved the way for the presentation of the TEE model for grades of membership by 1. Showing that there exist serious difficulties with present-day fuzzy set theory. 2. Identifying 14 different sources of fuzziness or uncertainty and showing how the first three of these (fuzziness #1a-3a) give rise to the grade of membership concept. 3. Defining LB (labeling), YN (yes-no) and MU (grade of membership) experiments, label sets, and natural language situations to which grade of membership functions refer. In addition, Hisdal (1988a) also sets up the first two assumptions of the TEE model which are summarized in points i)-iii) below.

i) The first assumption says that when a subject performs a semantic (LB or YN or MU) experiment under exact or nonexact conditions of observation, then her first step is to make an estimate u of the object's atribute value; e.g., an estimate of the object's height value when the experiment concerns a label set such as

$$\Lambda = \{\text{small, medium, tall}\} \;, \tag{1}$$

with linguistic height values. This estimate need not be a numerical one, it can be a comparison of the object's height with that of other, more familiar objects, e.g., the height of a door opening. (In many applied cases, u will be a point in a multidimensional universe.) The first assumption says that the answer which a subject gives in a semantic experiment is a function of the estimated attribute value u.

ii) The second assumption of the TEE model says that when a subject performs an exact YN or LB experiment in which she is acquainted with u^{ex}, the exact attribute value of the object, then she constructs nonfuzzy (but context dependent) quantization intervals Δu_λ in the universe U^{ex} such that she assigns the label λ to an object iff $u^{ex} \in \Delta u_\lambda$.

The following is a corollary of the first two assumptions:

iii) When a subject performs a nonexact LB or YN experiment, then she constructs nonfuzzy quantization intervals Δu_λ in the universe U of estimated attribute values such that she assigns the label λ to an object in an LB experiment iff $u \in \Delta u_\lambda$.

The second or 'threshold' assumption of the TEE model ('TEE' stands for 'Threshold', 'Error', 'assumption of Equivalence') concerns only LB and YN experiments. Such experiments exemplify natural language and everyday discourse situations in which a person says, e.g., "John is tall"; or in which she answers "Y" or "N" to the question "Is John tall?". In contrast, MU (grade of membership) experiments do not exemplify an everyday situation because numerical grade of membership values are not used in everyday discourse.

The present paper closes the circle by presenting the third or 'LB,YN–MU' assumption of Equivalence' which connects the numerical grade of membership values specified by a subject in a MU experiment with the answers given in an LB or YN experiment. The assumption formalizes the contention that there exists a positive correlation between the assignment of the label 'tall' to an object in an LB experiment, and the assignment to this object in a MU experiment of a high grade of membership value in the fuzzy set 'tall'.

The nonfuzzy quantization intervals used in LB or YN experiments according to the second assumption give rise to nonfuzzy, binary-valued $P(\lambda|u)$ 'threshold functions of u' which can assume solely the values 0 or 1. These functions are 'step'-shaped for extremal concepts λ (e.g., λ =old or λ =young), and they are 'square-pulse'-shaped for nonextremal concepts like 'middle-aged' (see fig. 2 and Hisdal 1988a).

We show in this paper in precise mathematical terms how these nonfuzzy LB or YN threshold curves in the universe U of estimated attribute values are converted to the S- or bell- shaped 'fuzzy threshold' or 'grade of membership' curves of fuzzy set theory in the universe U^{ex} of exact attribute values when the subject peforms an exact MU experiment. These washed-out threshold curves in U^{ex} are still more washed-out or fuzzified when the MU experiment is nonexact (see Hisdal 1986b, fuzziness #1b, section 3 and eqn (A9) in appendix. This last fuzzification effect was first discussed by Norwich & Turksen, 1982).

In order to limit the discussion, we shall deal mainly with fuzziness #1a in this paper; i.e. with fuzziness due to the subject's anticipation of errors of observation. Fuzziness #2a and intersubject fuzziness #3a have already been treated summarily in Hisdal (1986b).

Fig. 1 lists the main notation and terminology. A discussion of previous probabilistic interpretations of grades of membership is given in sect. 1.2. Sect. 2.1 shows how the nonfuzzy LB or YN threshold functions of U are converted to fuzzy LB or YN threshold functions of U^{ex}. The connection of these fuzzy threshold curves in U^{ex} with the membership functions elicited in a MU experiment is presented in sect. 2.2 through the LB,YN-MU assumption of equivalence. Experimental support for the TEE model is presented in sect. 2.3. The differentiation in the TEE model between distributions of $(\lambda|u^{ex})$ versus those of $(u^{ex}|\lambda)$ is presented in sect. 4 and illustrated in fig. 4. Eqn (21) of sect. 4 expresses $P(u^{ex}|\lambda)$ in terms of $\mu_\lambda(u^{ex})$ for an 'ideal subject', def. 3. The connection of the TEE model with previous formulas and concepts of fuzzy set theory is presented in sections 3 and 5. We conclude in sect. 6 that the TEE model has important consequences for the applications of fuzzy set theory; as well as for a theory of logic which is firmly founded on the basic metalanguage used by all human beings, namely natural language. Three important subjects have been relegated to the appendix in order not to interrupt the continuity of the paper. Appendix A1 presents an alternative LB,YN-MU assumption of equivalence and explains why we assign to it a minor role only. Appendix A2 uncovers the tight connection between Bandler and Kohout's checklist paradigm and the TEE model. Finally appendix A3 shows how the complementation paradox of fuzzy set theory is naturally resolved when the max operator for OR is replaced by the operators derived from the TEE model.

1.2 Probabilistic Interpretations of Grades of Membership and P

Zadeh (1978a) has made it quite clear that he considers possibility distributions of the attribute universe to be conceptually completely distinct from probability distributions. He has also made it clear that possibility distributions are numerically equal to grade of membership distributions (Hisdal, 1986a, eqn (1)). In the course of time, a number of other voices have, however, also made themselves heard.

Thus Hersh and Caramazza (1976) identify in their big experimental work the grade of membership of the object's exact attribute value u^{ex} in, e.g., the fuzzy set 'small', with the relative number of 'yes' answers concerning smallness of objects having that u^{ex} value.

Bandler & Kohout (1985) interpret partial truth values as the proportion of 'yes' answers checked off on a 'checklist'. We show in appendix A2 that the membership concept of the TEE model (see sect. 2 here) can be interpreted as a special application of Bandler & Kohout's checklist paradigm.

Lindley (1982) has investigated the question of possibilities versus probabilities from the standpoint of scoring rules. He concludes that only the $+$ and $\times$ operations are admissible, not the max and min operations. Natvig (1983) interprets possibility distributions as a family of probabilities and Saaty (1974) espouses a ratio scale for fuzzy sets.

Giles (1976, 1982) and Ruspini (1969) have both operated with probabilistic interpretations of grades of membership. Giles identifies grades of membership with subjective probabilities determined in a betting situation. And Ruspini, in his foundation laying 1969 paper on fuzzy clustering algorithms, says explicitly that he uses a probabilistic interpretation of grades of membership. Furthermore he sets up a formula according to which the grades of membership of an object in the different classes or clusters add up to 1. This formula has been retained both by himself, by Backer (1978), by Bezdec, Coray, Gunderson and Watson (1981), by Chaudhuri and Majumder (1982, p. 7, eqn (9)), and by Dunn (1974) in their subsequent work on fuzzy clustering algorithms. Which, by the way, are some of the most widely accepted fuzzy systems that we have today, also outside the fuzzy set community.

Gaines (1978, p. 167), suggests that membership values are averages over a population of binary 0 or 1 responses. And Kandel (1978, p. 1623) says: "Intuitively a similarity is felt between the concepts of fuzziness and probability. The problems in which they are used are similar or coincide. . . . . The fact that the assignment of a membership function of a fuzzy set is "nonstatistical" does not mean that we cannot use probability distribution functions in assigning membership functions." Zimmermann and Zysno (1980) say that they prefer the algebraic sum and the product operators to max and min.

In the following we attempt to summarize the points of similarity and difference between the TEE model and the probabilistic models of Gaines and

Giles, and Bandler & Kohout's checklist paradigm.

1. The TEE model contrasts the natural-language-exemplifying situation of an LB or YN experiment, in which the subject makes use of a linguistic label, with the situation of a MU experiment in which the subject answers with a numerical membership value chosen from the continuous interval [0,1] concerning such a label. It is only in the latter situation that we can talk of a partial grade of membership value according to the TEE model which identifies grade of membership values with the subject's estimate of the labeling- or Y-probability elicited in a great number of LB or YN experiments. This is in contrast to Gaines' and Bandler & Kohout's representations which do not make use of MU experiments at all, but operate solely with the averages obtained in YN experiments.

However, there is probably good agreement on this point between Giles' betting model and the TEE model. If a subject is willing to bet money concerning the correctness of a given statement (the correctness being ascertained by asking the 'first man in the street' for a YN answer, see Giles, 1976), then she is actually estimating the probability of occurrence of Y answers over all 'men in the street'. This interpretation also agrees with Giles' latest work in which he introduces the subject's 'degree of belief' (Giles 1988, sect. 3). A characteristic feature of Giles' work is that he does not operate at all with grade of membership functions of the attribute universe, only with grades of membership of objects. Giles concludes his 1988 paper with a pessimistic outlook for the grade of membership concept. This is probably due to his introduction of the more fuzzy 'homogeneous agents' and 'general agents' in addition to his 'Bayesian agent'. In our opinion Giles' pessimistic conclusion is not justified according to his own work. The existence of subjects who are so conscientious that they are averse to specifying an exact numerical membership value does not imply that such values are meaningless. We believe that it only means that such subjects estimate distributions or intervals over the [0,1] grade of membership interval in order not to bind themselves to a single value.

2. According to the first and second assumption of the TEE model, a sub-

ject who performs a YN or LB experiment has no other choice than that of basing her answer a) on her estimate of the object's attribute value and b) on the use of a nonfuzzy quantization interval in the universe of estimated attribute values. The last assumption agrees with Gaines' assumption of nonfuzzy thresholds. Gaines operates solely in the universe of exact attribute values. His, as well as Giles' membership values are to be identified with those of the TEE model obtained in an exact MU experiment, assuming that the subject operates solely with fuzziness #3 (intersubject fuzziness, see Hisdal, 1986b). Fuzziness #1 (variable conditions of observation, see present paper) and fuzziness #2 (representation in an underdimensioned universe, see Hisdal 1986b) are, as far as I understand these authors, not considered by Gaines and Giles.

The TEE model combines an operational definition of grades of membership in the form of MU experiments (also contained in Giles' betting model, though in a different form) with an interpretational definition which specifies a procedure that a subject can use in order to give her answer in an LB or YN experiment, or in order to specify the size of his bet (also contained in Gaines' work).

3. Finally the TEE model connects up these two definitions by the LB,YN-MU assumption of equivalence which interprets grades of membership as the subject's estimate of the labeling probability obtained in LB or YN experiments for objects of the same exact attribute value u^{ex}; the variability in labeling being due to fuzziness $\#1, 2$ or 3.

The TEE model thus attaches great significance to the complete definition of the meaning of the grade of membership concept. However, it also makes the following 'black box' assumption.

Definition 1 of the 'BLACK BOX' or fourth assumption of the TEE model. Often the detailed original meaning of the numerical grade-of-membership-values becomes gradually buried in a 'black box' in the subject's mind such that only the values of the membership functions for different (context dependent) concepts remain in her consciously accessable data base.

This assumption is in agreement with many other fields of artificial intelligence (e.g., visual pattern recognition or processing of language) in which we have learned that a very substantial part of human information processing proceeds on a subconscious level. It is just this 'black box effect' which makes artificial intelligence such a difficult and, at the same time, fascinating field.

2. The Meaning of Grades of Membership

2.1 Labeling Probabilities or Likelihood Functions of u^{ex}

Since grades of membership are connected up in subsection 2.2 with labeling or YN probabilities elicited in LB or YN experiments, we start by deriving a formula for these probabilities expressed in terms of 1) Δu_λ, the subject's quantization interval for λ in the universe U of estimated attribute values and 2) the real error function $P(u|u^{ex})$. This is the probability that the subject's estimate of the object's attribute value is u when the exact attribute value of the object (as determined by the experimenter in an exact experiment) is u^{ex}.

We start with the example shown in fig. 2. This presupposes a subject S for whom $u_{\lambda,l}$, the (lower) threshold value for 'tall man', in the universe U is 170 cm. Her nonfuzzy $t_{tall}(u)$ threshold curve for 'tall man' is shown by the step curve in the left half of fig. 2.

In general, the $t_\lambda(u)$ threshold curve for a concept λ is defined as a function of u which is equal to 1 inside the subject's quantization interval Δu_λ for λ, and to 0 outside this interval. According to the first and second assumptions of the TEE model, it can be interpreted as,

$$t_\lambda(u) = P(\lambda \mid u) \ , \tag{2}$$

the probability that the subject will assign the label $\lambda \in \Lambda$ (in an LB or YN situation) to an object whose attribute value she estimates to be equal to u.

Fuzzy set theory has always operated with membership functions of the exact attribute values u^{ex} of the objects, not of the estimated attribute values u (although u^{ex} is usually denoted by u in present-day theory). This is not only natural, but also necessary in an experimental situation because 1) u, the subject's estimate of the attribute value of the object, is unknown to the experimenter. 2) u^{ex} is an invariant for a given object, while u is not. In the following we therefore derive the shape of the $P(\lambda|u^{ex})$ curves from the shape of the nonfuzzy $P(\lambda|u)$ 'step' or 'square-pulse' curves. And we show that the former are a rounded-off or fuzzified version of the latter. More precisely, $P(\lambda|u^{ex})$ is the convolution of $P(\lambda|u)$ with the $P(u|u^{ex})$ error curve.

Before we start the derivation, we note that we assume a quantized attribute universe in our formulas and figures. For purposes of visualization, continuous curves are drawn through the discrete points of the figures, and subscripts on u are mostly left out in the formulas. The extension of the formulas to continuous universes is straightforward.

The lower and upper threshold values for the different elements λ_l of Λ are marked off in the figure as the midpoints between the greatest (upper) quantization point $u_{\lambda,u}$ of the concept $\lambda = \lambda_l$ to the left of the threshold, and the smallest (lower)quantization point $u_{\lambda',l}$ of the concept $\lambda' = \lambda_{l+1}$ to the right of the threshold (see Hisdal 1988a, item 5 of def. 1 and remark 2 of def. 11).

The derivation of the $P(\lambda|u^{ex})$ value for u^{ex} =175 cm is illustrated in fig. 2. The abcissa axis of this figure represents the *estimated* attribute-value u. The probability that the subject will label an object with exact attribute value u^{ex} as being λ is, according to our second assumption, equal to the probability that u will fall into S's quantization interval $\Delta u_\lambda = \{u_{\lambda l}, \ldots, u_{\lambda u}\}$ for λ,

$$P(\lambda|u^{ex}) = \sum_{u=u_{\lambda l}}^{u_{\lambda u}} P(u|u^{ex}) = \sum_{u=-\infty}^{\infty} t_\lambda(u)\ P(u|u^{ex})\ , \tag{3}$$

where $t_\lambda(u)$ is the nonfuzzy 'threshold curve' (see illustrations in fig. 2 for λ equal to 'tall' and 'medium' respectively).

The broken curve in fig. 2 shows an assumed $P(u|u^{ex})$ real error function for u^{ex} =175 cm. $P(\lambda|u^{ex})$ is equal to the sum of the ordinates of this function in the shaded areas for λ =tall and λ =medium respectively. For a each value of u^{ex}, we must now displace the error curve to $u = u^{ex}$ and compute the sum (3). This results in the fuzzy or rounded $P(\text{tall}|u^{ex})$ threshold curve for 'tall' of fig. 3(b). The nonfuzzy threshold curve in the same figure can be interpreted as the $P(\text{tall}|u^{ex})$ curve elicited in an *exact* YN experiment for which $u = u^{ex}$.

Note that there is nothing fuzzy about the *meaning* of the fuzzy $P(\lambda|u^{ex})$ threshold curves. The value of $P(\lambda|u^{ex})$ denotes the probability that an object with exact attribute value u^{ex} will be assigned the label $\lambda \in \Lambda$ by the subject; the uncertainty in labeling being due to errors of estimation of u^{ex}.

In statistical terminology the $P(\lambda|u^{ex})$ labeling probabilty is called a likelihood distribution over the conditioning variable u^{ex}. We will therefore also use the alternative names 'likelihood distribution of λ over u^{ex}' or 'fuzzy threshold function of u^{ex} for the label λ' for this quantity.

We can sum up the results of this section by stating that, for a given subject, the labeling probability $P(\lambda)$ is a nonfuzzy threshold function of u, the estimated attribute value of the object (see 'step'- and 'square pulse' curves of fig. 2). When $P(\lambda)$ is considered to be a function of u^{ex}, the exact attribute value of the object, then eqn (3) shows that it is converted to a fuzzy threshold function (see rounded curve in fig. 3(b)). The nonfuzzy function of u and the fuzzy function of u^{ex} refer to the same YN or LB experiment. (By a nonfuzzy and fuzzy function we mean a function whose range is $\{0, 1\}$ and [0,1] respectively.)

The bigger the width of the error curve (fig. 3(a)), the bigger is the fuzzification or rounding-off effect. For an exact YN or LB experiment we have that $u = u^{ex}$, and consequently the error curve has the width 0 (i.e., it is a delta function). In this case there is no fuzzification effect, and the labeling probability is given by a nonfuzzy threshold function not only of u, but also of u^{ex}. This function is shown by the step curve in fig. 3(b).

2.2 The LB,YN-MU Assumption of Equivalence

In this subsection we finally make the important connection between the results of LB or YN experiments on the one hand, and those of MU experiments on the other. The connecting link is the third or LB,YN-MU assumption of equivalence of the TEE model, def. 2 below.

We shall assume that the grade of membership curves refer to an exact MU experiment performed by the subject. (For nonexact MU experiments, see Hisdal 1986b, sect. 1 and appendix, fuzziness #1b.)

As a preliminary, we start with our λ =tall example. In an exact YN or LB experiment, the $P(\lambda|u^{ex})$ curve is now the *nonfuzzy* threshold curve of fig. 3(b).

In a MU experiment, the subject is no longer required to select a label $\lambda \in \Lambda$, or to give a YN answer concerning the object's being λ. Instead she is instructed to assign to an object with a given u^{ex}-value a grade of membership-value in the class 'tall' (see Hisdal 1988a, defs. 4, 5). We know that she then performs some sort of smoothing operation on this step curve. The TEE model now says, that the original meaning of the ordinates of this smoothed grade of membership curve is the following:

When asked to what degree a person is tall, the subject (who knows that the object's exact height is 175 ± 5 cm) puts herself into the situation of everyday life in which she cannot measure the exact height value of each object. She knows that under such conditions she will make errors of observation. And she takes account of this knowledge by constructing an estimated error curve $P^{est}(u \mid (u^{ex}{=}175 \text{ cm}))$ and saying to herself: "Under everyday conditions of observation, I would estimate the object's height to lie in my quantization interval for 'tall' in 75% of all cases. In 25% of all cases I would estimate it to lie in my quantization interval for 'medium', and would therefore not assign the label 'tall' to the object. The grade of membership of this object in the class 'tall' is therefore

$$\mu_{tall}(u^{ex}{=}175 \text{ cm}) = P(\text{tall} \mid (u^{ex}{=}175 \text{ cm})) = 0.75 \text{ ."} \tag{4}$$

This is the situation for u^{ex} =175 cm depicted in figs. 2, 3, assuming that the error curve $E(x)$ in fig. 3(a) is the subject's estimated error curve $P^{est}(x|u^{ex})$, where $x = u - u^{ex}$. In general the subject carries out this operation for every value of u^{ex}, thereby converting the nonfuzzy threshold curve of fig. 3(b) to the fuzzy threshold or grade of membership curve of that figure. Finally the subject stores this membership curve in her knowledge base, its original meaning becoming a 'black box' whose contents may be forgotten. The 'black box assumption' has already been stated in def. 1, end of sect. 1.2.

The following is a more formal statement of the LB,YN-MU assumption.

Definition 2 of the LB,YN-MU assumption of equivalence or the third assumption of the TEE model for fuzziness #1a (for #2a and 3a, see Hisdal (1986b, sect. 2.3 and appendix)). When a subject performs a MU experiment under exact conditions of observation, she puts herself into the situation of an observation under nonexact conditions. Her grade of membership curve is her estimate of the modification of her nonfuzzy LB or YN threshold curve by the error curve. The word 'estimate' in this connection referring both to her estimate of the probabilities of error and to her estimate of the mathematically computed effect of these errors in rounding-off the nonfuzzy threshold curve,

$$\begin{aligned}\mu_\lambda^{excond}(u^{ex}) = {} & \text{ subject's estimate of } P(\lambda|u^{ex}) \text{ under nonexact conditions} \\ = {} & P^{est-nexcond}(\lambda|u^{ex}) \text{ .}\end{aligned} \tag{5}$$

The superscripts 'excond' and 'est-nexcond' on the left and right hand sides of eqn (8) refer to a membership and a likelihood function elicited under 'exact conditions of observation' and 'nonexact conditions of observation' respectively.

The latter being the estimated nonexact conditions assumed by the subject in connection with her fuzziness #1a.

The value of $P^{est-nexcond}(\lambda \mid u^{ex})$ on the right hand side of eqn (5) is found from eqn (3), except that we must now replace the real error curve $P(u|u^{ex})$ by $P^{est}(u|u^{ex})$, the subject's estimate of this curve. Assuming that the subject is able to carry out the mathematical operation of eqn (3) correctly, we have then

$$\mu_{\lambda}^{excond}(u^{ex}) = \sum_{u=u_{\lambda l}}^{u_{\lambda u}} P^{est}(u|u^{ex}) = \sum_{u=-\infty}^{\infty} t_{\lambda}(u)\ P^{est}(u|u^{ex}) \ , \tag{6}$$

where $t_{\lambda}(u)$ is the nonfuzzy threshold curve, see fig. 2. The subject will usually refer her membership curve to everyday conditions of observation; i.e., $P^{est}(u|u^{ex})$ is her estimate of the error curve under everyday conditions. Her membership curve for λ, as elicited in a MU experiment performed under exact conditions, will then be equal to her likelihood curve for λ, as elicited under everyday conditions; provided that her estimate of the error curve of everyday conditions, and her computation of the effect of these errors on the nonfuzzy threshold or likelihood curve is correct.

2.3 Trying to Falsify the TEE model

An outline of the TEE model was first presented in Las Palmas (Hisdal 1982), and was immediately criticized by I.B. Turksen on the ground that his experimental results showed that MU experiments result in wider curves than YN experiments (Norwich & Turksen, 1982). Already then I could tell Turksen that this is just what is to be expected from the TEE model when the two experiments are performed under the same conditions of observation. Indeed, if Norwich and Turksen had found that the two curves are approximately identical, then this result would have been a falsification of the TEE model. This subsection is devoted mainly to a discussion of this point.

Later, Turksen (private communication) criticized the TEE model on the ground that an experimental test of eqn (5) would require the YN experiment and the MU experiment to be performed under different conditions of observation while "all the psychophysical literature is in favor of comparing two functions under identical conditions".

This criticism is not really to the point. If the subject's meaning of the membership function is indeed that of a tool used for communicating her estimate of the effect of nonexact conditions of observation on the labeling of objects, then we cannot reject eqn (5) just because it is inconvenient from an experimental point of view.

Finally Turksen has criticised the notion of 'everyday conditions of observation' because of the difficulty of defining and applying such conditions. Again his attack on the TEE model is due to the supposedly great experimental difficulties for testing it. Assuming for the moment that these difficulties are real, they would not be a sufficient ground for pronouncing a theory as being false. E.g., Einstein's prediction of the dependence of time intervals on the movement of the coordinate system in which they are measured was only verified experimentally decennia later. But the theory of relativity was not falsified in 1906 because this prediction could not be verified with the technologies and instrumentations available at that time. (For possible definitions of 'everyday conditions of observation', see Hisdal 1984a section 5 and definition 5.14; also Hisdal 1986b, sect. 1.)

In this subsection we show that it is not necessary to perform a YN and a MU experiment under different conditions of obseervation in order to test the TEE model. Norwich and Turksen's experimental setup of a YN and a MU experiment performed under the same conditions of observation *can* be used as a partial test of this model. Furthermore, a good definition of everyday conditions of observation is not a prerequisite for the TEE model. All that is required, is that the subject make some assumption about the frequency of errors of observation which occur under the uncontrolled conditions of everyday life.

In general we would not expect that the subject can carry out the computational part of eqn (6) exactly when she performs an exact MU experiment, or when she stores her internal membership function. The better the agreement between $\mu_\lambda^{excond}(u^{ex})$ and the right hand side of eqn (6), the more consistent is the subject's information processing in connection with grade of membership assignments. To formalize this statement, we define an ideal subject as follows.

Definition 3 of an ideal subject. This is a subject who

1. Consistently uses the same estimated error curve $P^{est}(u|u^{ex})$ in connection with MU experiments.

2. Consistently uses the same lower and upper thresholds in u in semantic experiments concerning λ, and referring to the same situation (concerning the situation dependence, see Hisdal 1988a, sections 1 and 6).
3. Performs the summation operation in eqn (6) correctly for all u^{ex}.

In summary, an ideal subject always uses the same internal $\mu_\lambda(u)$ membership function in a MU experiment referring to a given label and a well-defined situation. In an exact MU experiment, the ordinates of this function of $u = u^{ex}$ are given by eqn (6). In a nonexact MU experiment, $\mu_\lambda(u^{ex})$ is not a constant. Its expectation value is given in Hisdal (1986b eqn (A9); see also eqns (A7), (A8) concerning the internal membership function).

The formulas of the TEE model papers which involve grades of membership assume an ideal subject unless something else is mentioned.

We are now ready to say something about the relation between $\mu_\lambda(u^{ex})$ and $P(\lambda|u^{ex})$ when both are elicited from an ideal subject under the same conditions of observation. Suppose first, that these are exact conditions. According to the second assumption of the TEE model, $P^{excond}(\lambda|u^{ex})$ is then given by the nonfuzzy $t_\lambda(u)$ threshold curve of figs. 2 and 3(b). While $\mu_\lambda^{excond}(u^{ex})$ is, according to eqns (5), (6) and figs. 2, 3, a rounded-off version of the nonfuzzy threshold curve.

Suppose now that $P(\lambda|u^{ex})$ and $\mu_\lambda(u^{ex})$ are both elicited under the same nonexact conditions, namely those assumed by the subject in connection with her fuzziness #1a. $P^{nexcond}(\lambda|u^{ex})$ is then identical with the right hand side of eqn (6) (and with the rounded curve in fig. 3(b)). While the expectation of $\mu_\lambda^{nexcond}(u^{ex})$ (over objects of a given u^{ex}) is a rounded version of $\mu_\lambda^{excond}(u^{ex})$, the left hand side of (6) ($\mu_\lambda^{excond}(u^{ex})$ being identical with the rounded curve of fig. 3(b)). More precisely, $\mu_\lambda^{nexcond}(u^{ex})$ is the convolution of $\mu_\lambda^{excond}(u^{ex})$ and the real error curve of the nonexact conditions (see Hisdal 1986b, eqn (A9) in appendix). We have thus the following result:

Theorem 1. The LB,YN-MU Theorem. When a membership curve $\mu_\lambda(u^{ex})$ and an LB or YN likelihood curve $P(\lambda|u^{ex})$ are both elicited under the same, real conditions of observation, these being either exact conditions, or the nonexact conditions estimated by the subject in connection with her fuzziness #1a, then the expected membership curve is a rounded-off version of the likelihood curve. More precisely, it is the convolution of the likelihood curve with the estimated error curve.

Norwich and Turksen's YN and MU experiment were performed under the same conditions of observation (although these probably lie in between exact and everyday conditions), and they find just this qualitative result. Indeed Norwich and Turksen (private communication) write: "Let us consider the two types of experiment under identical conditions, regardless of whether they are 'everyday' or 'exact'. As described in the discussion following Theorem 2 (in Norwich & Turksen 1982) and culminating in Theorem 3, the LB,YN and MU experiments are not equivalent under any identical conditions, which we may denote 'C_i'. In Hisdal's notation then our Theorem 3 means that

$$P^{\mathrm{YN}\ (conditions\ C_i)}(\lambda|u^{ex}) \neq \mu_\lambda^{(conditions\ C_i)}(u^{ex}) \ . \tag{7}$$

Moreover, we have performed this comparison empirically hundreds of times and equality has never occurred. *The size of the fuzzy region of* μ_X *is typically many times that of the fuzzy region of* P^{YN} *.*" (The last sentence is underlined in Norwich and Turksen's communication.)

The fact that Norwich & Turksen find exactly the qualitative result predicted by the TEE model does not, of course, prove the correctness of this model. We know from Popper's work (Popper, 1969) that a theory can never be proved experimentally, it can only be falsified. We *can* say, however, that if one assumes that the TEE model is correct, then one finds that it predicts just the experimental result of Norwich and Turksen.

We remark that for fuzziness #2a and 3a we have similar YN-MU assumptions of equivalence (see appendix of Hisdal 1986b for details). For fuzziness #3a, the subject performing the MU experiment puts herself into the role of other subjects, realizing that these may have thresholds for 'tall' which differ somewhat from her own. Resulting again in a rounded-off or fuzzy threshold curve.

When two or all three sources of fuzziness are present simultaneously, then the rounded-off likelihood or grade of membership curves due to one source of fuzziness are further rounded-off by the other sources.

3. Some Consequences of the TEE Model

We shall here derive eight consequences of the first three assumptions of the TEE model. These consequences resolve various former difficulties in fuzzy set theory (listed under difficulties 7 and 14-16 in Hisdal 1986a).

The first three consequences are theorems which follow from the TEE model and which have, up to now, been postulates of fuzzy set theory. These theorems concern the S- and bell- shapes of the membership curves, the 'one minus' theorem for the negation and the 'summing up to one' theorem of fuzzy clustering algorithms. The remaining consequences clarify the meaning of previously-used concepts or numerical values, or formulas; namely the meaning of the 0, 0.5 and 1 values of grade of membership functions, of a 'subnormal' fuzzy set, and of the square of the membership function for λ. Further consequences are derived in sect. 5. In contrast to the consequences of the present section, those of sect. 5 depend also on the prior distibution $P(u^{ex})$.

Zadeh (1976 p. 256; 1977, p. 10; 1978a, pp. 5,6; 1978b, p. 404) postulates that the $\mu_\lambda(u^{ex})$ grade of membership functions are S- or bell- shaped. In the TEE model we have the following corresponding consequence 1 which is derived from eqn (6) and from fig. 2. These show that the membership functions of extremal concepts are cumulative probability functions.

Consequence 1 concerning the shapes of the membership functions. The membership functions of upper extremal concepts like 'tall' or 'VERY tall' are non-decreasing functions of u^{ex}. Those of lower extremal concepts like 'small' or 'VERY small' are non-increasing functions; and those of non-extremal concepts like 'medium' (or 'tall' with LB reference, assuming that 'VERY tall' is an element of the reference label set, see Hisdal 1988a, sect. 1 and sect. 6, def. 15) are unimodal functions (i.e., functions with a single hump). Assuming a unimodal error function $P^{est}(u|u^{ex})$, the above three shapes reduce to the previously *postulated* S, 1–S and bell shapes respectively.

Zadeh (1973, p. 32) postulates that the grade of membership of 'NOT λ' is equal to one minus the grade of membership of λ for the same u^{ex}. To derive the corresponding *theorem* of the TEE model we start with the following theorem derived in Hisdal (1988a, sect. 5, theorem 3):

$$P(\lambda \mid u^{ex}) + P(\text{NOT-}\lambda \mid u^{ex}) = 1 \qquad \forall u^{ex}, \tag{8}$$

where the first and second terms on the left hand side are the probabilities of a Y and N answer respectively to the question "Is this object λ?". This theorem is simply a consequence of the requirement that in a YN experiment the subject must answer either 'Y' or 'N'.

In Hisdal (1988a, appendix A1) we show that natural language makes use of many other, situation-dependent interpretations of the negation all of which are, however, built on top of the above simple interpretation referring to a straightforward YN experiment.

Assuming an ideal subject we derive the following consequence 2 from (8) and the LB,YN-MU assumption.

Consequence 2. Derivation of the 'one minus' formula for the negation. It follows from the TEE model that the following equation holds for the grades of membership elicited in an exact MU experiment with YN reference,

$$\mu_\lambda(u^{ex}) + \mu_{NOT\,\lambda}(u^{ex}) = 1 \qquad \forall u^{ex} \ , \tag{9}$$

where the label 'NOT λ' refers to a YN-MU experiment concerning NOT λ =NOT λ_{spec}. (See defs. 3, 5 in Hisdal 1988a. In appendix A1 of that paper it is argued that the direct use of negated specified labels in formal YN and YN-MU experiments is not to be recommended.)

Eqn (9), combined with the traditional max operator for the union of fuzzy sets, results in the complementation paradox of fuzzy set theory. In appendix A3 we show that this paradox disappears in the TEE model if we define the union of two fuzzy sets a, b as the fuzzy set 'a *OR* b'; provided that we *derive* the operation for the inclusive OR connective instead of *postulating* it to be the max operation. It then turns out that the membership function for 'a *OR NOT* a' has the value 1 for all u^{ex}.

In sect. 1.2 we have already mentioned the 'summing up to one' formula used in all fuzzy clustering algorithms. In the TEE model we start out with theorem 2 of Hisdal (1988a, sect. 5) which says that in an LB experiment, the sum of $P(\lambda_l|u^{ex})$ over all $\lambda_l \in \Lambda$ is equal to 1. From this theorem and the LB,YN-MU assumption we derive the following consequence 3.

Consequence 3. Derivation of the 'Summing up to 1' formula. It follows from the TEE model that the following formula holds for the grades of membership elicited in an exact MU experiment with LB reference (Hisdal 1988a, sect. 2, def. 4),

$$\sum_{l=1}^{L} \mu_{\lambda_l}(u^{ex}) = 1 \qquad \forall u^{ex} . \tag{10}$$

Eqns (9) and (10) are also valid when u^{ex} is replaced by u, irrespective of whether the MU experiment is exact or not. For a nonexact MU experiment, the two equations are valid when all $\mu(u^{ex})$ functions are replaced by their expected values (with respect to the $\mu \in [0,1]$ values assigned in the nonexact MU experiment to objects of a given u^{ex}, see Hisdal 1986b, p. 134.) The common reason for all these equations, which are valid for an ideal subject, def. 3, is that the subject refers her membership values to the natural language situation of an LB or YN experiment; and that she must necessarily assign one of the $\lambda_l \in \Lambda$ labels, or one of the two YN values, in such an experiment.

In Hisdal (1986a, sect. 3, difficulty 1) we discussed the unsatisfactory situation in the present-day theory of possibility which does not allow us to distinguish between a certainty and a mere possibility. In the TEE model this difficulty is clarified through the well-defined meaning of a $P(\lambda|u^{ex}) = 1$ value and the $\mu_\lambda(u^{ex}) = 1$ value associated with it through the LB,YN-MU assumption of equivalence. Furthermore, the second or threshold assumption of the TEE model allows us to fix precisely the subset of the U^{ex} domain for which the membership function has the value 1. This is done by drawing figures analogous to fig. 2, but with the $u = u^{ex}$ point of the error function displaced successively to all points on the u-axis for which the complete error function lies inside the quantization interval for λ. Similar statements hold in connection with the grade of membership values 0 and 0.5. These three cases are stated in consequences 4, 5 and 6 below.

Consequence 4. The meaning of the u^{ex} values for which $\mu_\lambda(u^{ex}) = 1$. Let a subject perform an exact MU experiment with an object of exact attribute value u^{ex}. Since the subject interprets $\mu_\lambda(u^{ex})$ as $P^{est}(\lambda|u^{ex})$, she will assign the membership value 1 to the object iff she estimates that she would *always* assign the label λ to this object in an LB or YN experiment, irrespective of the point of the set of conditions of observation under which it is observed. This happens for the following values of u^{ex} ,

$$u_{\lambda,l} + w_{left}(u^{ex}) \leq u^{ex} \leq u_{\lambda,u} - w_{right}(u^{ex}). \tag{11}$$

In (11), $u_{\lambda,l}$, $u_{\lambda,u}$ are the lower and upper bounds respectively of the subject's quantization interval Δu_λ for λ . w_{left} and w_{right} (see fig. 3(a)) are the size of the u regions to the left and right of $u = u^{ex}$ for which $P^{est}(u|u^{ex}) > 0$.

We see that the smaller the width of the estimated error curve in relation to the size $\delta u_\lambda = u_{\lambda,u} - u_{\lambda,l}$ of the quantization interval Δu_λ for λ , the bigger is the U^{ex} region for which $\mu_\lambda(u^{ex}) = 1$. The region is biggest for a subject who assumes an error curve with width 0, such that her estimate u is always equal to u^{ex} . This results in a nonfuzzy, subjective grade of membership curve which coincides with the nonfuzzy threshold curve.

Consequence 5. The meaning of the u^{ex} values for which $\mu_\lambda(u^{ex}) = 0$. Let a subject perform an exact MU experiment with an object of exact attribute value u^{ex} . Since the subject interprets $\mu_\lambda(u^{ex})$ as $P^{est}(\lambda|u^{ex})$, she will assign the membership value 0 to the object iff she estimates that she would *never* assign the label λ to this object in an LB or YN experiment, irrespective of the point of the set of conditions of observation under which it is observed. This happens when u^{ex} satisfies one of the two conditions below,

$$u^{ex} \leq u_{\lambda,l} - w_{right}(u^{ex}), \qquad \text{or} \qquad u^{ex} \geq u_{\lambda,u} + w_{left}(u^{ex}) \ . \tag{12}$$

The bigger the width of the estimated error curve in relation to the size of the quantization interval for λ , the smaller are the U^{ex} regions for which $\mu_\lambda(u^{ex}) = 0$. For an infinitely wide error curve there exists no u^{ex} for which $\mu_\lambda(u^{ex}) = 0$.

Zadeh (1973, p. 30) defines the 'crossover points' of $\mu_\lambda(u^{ex})$ as those values of u^{ex} for which $\mu_\lambda(u^{ex})$ =0.5. In the TEE model it can be shown that the crossover points coincide with the subject's threshold value(s) for λ under certain, not too restrictive, conditions. The name 'crossover points' is thus very fitting one according to the TEE model.

Consequence 6. The connection between the crossover points and the threshold values. This connection is best stated in the form of the following theorem.

When u and u^{ex} take on values in a continuous universe, then the crossover points of $\mu_\lambda(u^{ex})$ coincide with the subject's threshold values for λ (as elicited in an LB or YN experiment),

$$\mu_\lambda(u^{ex} = u_{\lambda,l}) = 0.5 \ , \tag{13}$$

$$\mu_\lambda(u^{ex} = u_{\lambda,u}) = 0.5 \ , \tag{14}$$

under the following conditions: (13) holds if conditions (13a) and (13b) below are satisfied; and (14) holds if conditions (14a) and (14b) are satisfied.

Conditions (13a) and (14a) say that the median of the $P^{est}(u|u^{ex})$ curve must, for $u^{ex} = u_{\lambda,l}$ and $u^{ex} = u_{\lambda,u}$ respectively, coincide with the point $u = u^{ex}$,

$$\int_{-\infty}^{u^{ex}} P(u \mid u^{ex} = u_{\lambda,l}) \ du = \int_{u^{ex}}^{\infty} P(u \mid u^{ex} = u_{\lambda,l}) \ du = 0.5 \ , \tag{13a}$$

$$\int_{-\infty}^{u^{ex}} P(u \mid u^{ex} = u_{\lambda,u}) \ du = \int_{u^{ex}}^{\infty} P(u \mid u^{ex} = u_{\lambda,u}) \ du = 0.5 \ . \tag{14a}$$

Conditions (13b), (14b) require that w_{left} and w_{right} respectively (see fig. 3(a)) must not exceed the size of the quantization interval for λ,

$$w_{right}(u^{ex} = u_{\lambda,l}) \leq \delta u_\lambda = u_{\lambda,u} - u_{\lambda,l} \ . \tag{13b}$$

$$w_{left}(u^{ex} = u_{\lambda,u}) \leq \delta u_\lambda = u_{\lambda,u} - u_{\lambda,l} \ . \tag{14b}$$

The next consequence concerns the mystic concept of a subnormal fuzzy set, i.e., a fuzzy set whose biggest membership value is smaller than 1. We have already discussed Norwich and Yao's and Norwich and Turksen's 'brutal' device of demystifying a subnormal fuzzy set by normalizing it such that its biggest and smallest membership values are 1 and 0 respectively (Hisdal 1986a, difficulty 16b; 1986b, fuzziness #2a in appendix).

According to the TEE model, subnormality occurs in connection with fuzziness #1 when there exists no value of u^{ex} for which the subject estimates that she is certain to assign the label λ to the object in an LB or YN experiment under any condition of observation (belonging to the set of conditions of observation to which she refers her fuzziness #1).

Said in another way, subnormality occurs when the width of the subject's estimated $P^{est}(u|u^{ex})$ error curve is bigger than the quantization interval for λ

(for a more precise formulation, which takes into account that the error curve may depend on u^{ex}, see consequence 7 below). Thus subnormality is demystified in the TEE model not by replacing the subnormal fuzzy set by a normal one, but by a clear differentiation in meaning between a normal and a subnormal fuzzy set.

Consequence 7 concerning subnormality of $\mu_\lambda(u^{ex})$. The membership function $\mu_\lambda(u^{ex})$ is subnormal iff there exists no value of u^{ex} for which eqn (11) is satisfied. It follows from eqn (11), that an equivalent condition is that the width of the error curve must always be bigger than the quantization interval for λ. I.e., $\mu_\lambda(u^{ex})$ is subnormal iff $w > \Delta u_\lambda \ \forall u^{ex}$, where $w = w(u^{ex}) = w_{left}(u^{ex}) + w_{right}(u^{ex})$ is the size of the u region for which $P^{est}(u|u^{ex}) > 0$ (see fig. 3a).

In 'traditional' fuzzy set theory, the membership function for 'VERY λ' is defined as the square of the membership function for λ (Zadeh 1973, eqn (3.3)).

We have already discussed in Hisdal (1988a, sect. 6 and figs. 4,5; 1986a, difficulties 7 and 14b) that this representation of 'VERY λ' is not satisfactory, and that a displacement of the $\mu_\lambda(u^{ex})$ curve along the u^{ex} axis is a better representation of the membership function of 'VERY λ' both according to experimental results and according to the TEE model. However, $\mu_\lambda^2(u^{ex})$ does have a definite meaning in the TEE model as stated in consequence 8 below.

Consequence 8. $\mu_\lambda^2(u^{ex})$ represents 'λ AND λ' (with RR reference), not 'VERY λ'. Let $\mu_\lambda(u^{ex})$ be the subject's membership function for λ. Then the function $\mu_\lambda^2(u^{ex})$ is equal to $\mu_{\lambda\ AND\ \lambda}(u^{ex})$, her membership function for 'λ AND λ', provided that she refers the AND connective to an RR situation. This means that the grade of membership which the subject assigns to an object in 'λ AND λ' is equal to her estimate that she will assign the label λ to the object in two successive LB or YN experiments which are such that the objects are rerandomized with respect to the points of the set of conditions of observation between the two experiments.

The theorem of consequence 8 is proved in Hisdal (1984a, eqns (10.37), (12.18)). The subject of the connectives will be treated in detail in (Hisdal, 1988b).

4. Quantities which Depend on the Prior Distribution

4.1 λ-Qualified Probabilities and Related Quantities

In this section we treat three quantities which depend on the 'prior' or 'unqualified' distribution $P(u^{ex})$ (e.g., the distribution over height of the population of men, unqualified by a label such as 'λ=tall man' or 'λ=small man'). These are the λ-qualified probability distribution $P(u^{ex}|\lambda)$, the marginal labeling probability $P(\lambda)$, and the RR 'autological probability' $P(\lambda|\lambda)$. The latter two are identified in section 5 with Zadeh's probability of the fuzzy event λ, and with his possibility/probability consistency γ respectively. Furthermore the entropy of a fuzzy set is discussed in sect. 5, as well as the resolution of two previous difficulties.

The present subsection refers to LB or YN experiments for the assignment of the label λ. Grades of membership are introduced into the formulas in subsection 4.2. Subsection 4.3 discusses the prior distribution which pertains to a given situation.

In contrast to the likelihood distribution $P(\lambda|u^{ex})$ of sect. 2.1 which denotes the probability that an object of attribute value u^{ex} will be assigned the label λ, $P(u^{ex}|\lambda)$ denotes the probability that an object which has been assigned the label λ in a YN or LB experiment has the attribute value u^{ex}.

The numerical relation between $P(u^{ex})$, $P(\lambda|u^{ex})$ and $P(u^{ex}|\lambda)$ is easily found from the law of compound probabilities,

$$P(x_i, y_j) = P(x_i)\ P(y_j|x_i) = P(y_j)\ P(x_i|y_j)\ . \qquad (15)$$

In (15), x_i and y_j are the values of two attributes X and Y of an outcome of a statistical experiment. In our case the attributes refer to the linguistic label and the exact attribute value assigned to a single object by the subject and the experimenter respectively. Eqn (15) is valid irrespective of whether X and Y are statistically dependent or not.

The 'marginal probability' $P(y_j)$ in (15) is given by

$$P(y_j) = \sum_i P(x_i, y_j) = \sum_i P(x_i)\ P(y_j|x_i)\ . \qquad (16)$$

From (15) it follows that

$$P(x_i|y_j) = P(x_i)\ P(y_j|x_i)\ /\ P(y_j)\ , \qquad (17)$$

where $P(y_j)$ is given by (16).

Substituting u_i^{ex} for x_i and λ_l for y_j in (17) (we abbreviate these two to u^{ex} and λ respectively), we obtain

$$P(u^{ex}|\lambda) = P(u^{ex})\ P(\lambda|u^{ex})\ /\ P(\lambda)\ , \tag{18}$$

where the marginal labeling probability $P(\lambda)$ is, according to (16), given by

$$P(\lambda) = \sum_{u^{ex}} P(u^{ex}, \lambda) = \sum_{u^{ex}} P(u^{ex})\ P(\lambda|u^{ex})\ . \tag{19}$$

Finally we compute the 'autological probability' $P(\lambda|\lambda)$ referring to an RR composite experiment. This is the probability that the label $\lambda \in \Lambda$ will be assigned to an object in an LB or YN experiment when it has been assigned to the same object in a previous such experiment; assuming that the object is ReRandomized with respect to the points of the set of conditions of observation between the two experiments. (See Hisdal 1988b or Hisdal 1984a, sect. 10.4 for RR composite experiments; and sect. 11 of the last reference for 'autological probabilities'.)

$P(\lambda|\lambda)$ is equal to 1 in the absence of fuzziness. In general it is given, for RR reference, by the formula

$$P^{RR}(\lambda|\lambda) = \sum_{u^{ex}} P(\lambda|u^{ex})\ P(u^{ex}|\lambda) = \sum_{u^{ex}} P(u^{ex})\ P^2(\lambda|u^{ex})\ /\ P(\lambda)\ , \tag{20}$$

where the expression after the last equality sign is obtained by substituting for $P(u^{ex}|\lambda)$ from (18).

The following is a summary of the results of this subsection. $P(\lambda)$, the unqualified or marginal labeling probability for λ, is given by eqn (19), and the autological probability $P(\lambda|\lambda)$ for RR reference by (20). From (18) and (19) it follows that the $P(u^{ex}|\lambda)$ function of u^{ex}, eqn (18), is equal to the normalized product of the $P(u^{ex})$ and $P(\lambda|u^{ex})$ functions of u^{ex}.

4.2 Introducing Grades of Membership into the Formulas

We shall here make use of the LB,YN-MU assumption of equivalence, def. 2, and substitute $\mu_\lambda(u^{ex})$ (as elicited in a MU experiment) for $P(\lambda|u^{ex})$ (as elicited in an LB or YN experiment) into eqns (18)-(20). The resulting formulas for $P(\lambda)$, $P(u^{ex}|\lambda)$ and $P(\lambda|\lambda)$ are then expressions for probabilities computed on the basis of the subject's estimate $\mu_\lambda(u^{ex})$ of the qualified labeling probability $P(\lambda|u^{ex})$ which would be elicited in an LB or YN experiment in the presence of fuzziness. The connection of these formulas with quantities which have previously been used in the theory of fuzzy sets, and the significance of the formulas for the resolution of some previous inconsistencies are discussed in sect. 5.

Making the substitution of eqn (5) into (18), (19) and (20) we obtain,

$$P(u^{ex}|\lambda) = P(u^{ex})\ \mu_\lambda(u^{ex})\ /\ P(\lambda)\ , \tag{21}$$

$$P(\lambda) = \sum_{u^{ex}} P(u^{ex})\ \mu_\lambda(u^{ex})\ , \tag{22}$$

$$P^{RR}(\lambda|\lambda) = \sum_{u^{ex}} \mu_\lambda(u^{ex})\ P(u^{ex}|\lambda) = \sum_{u^{ex}} P(u^{ex})\ \mu_\lambda^2(u^{ex})\ /\ P(\lambda)\ . \tag{23}$$

4.3 The Prior Distribution

Each of the three distributions (18), (19), (20), and its corresponding expression (21), (22), (23) in terms of the membership function, is consistent in the sense that

$$\sum_{u^{ex}} P(u^{ex}|\lambda_{l'}) = \sum_{l=1}^{L} P(\lambda_l) = \sum_{l=1}^{L} P(\lambda_l|\lambda_{l'}) = 1 \qquad \forall l'\ , \tag{24}$$

provided that the $P(\lambda|u^{ex})$ (or $\mu_\lambda(u^{ex})$) and $P(u^{ex})$ functions are consistent, i.e. provided that

$$\sum_{l=1}^{L} P(\lambda_l|u^{ex'}) = \sum_{l=1}^{L} \mu_{\lambda_l}(u^{ex'}) = \sum_{u^{ex}} P(u^{ex}) = 1 \qquad \forall u^{ex'}\ . \tag{25}$$

In (24), (25), λ_l and $\lambda_{l'}$ are elements of $\Lambda = \{\lambda_l\}$, $l = 1, \ldots, L$, the reference label set for the semantic experiment.

There exist many prior distributions $P(u^{ex})$ which satisfy the consistency requirement $\sum_{u^{ex}} P(u^{ex}) = 1$. How do we choose the correct one to insert into eqns (18)-(23)?

As in so many other cases, the answer to this question depends on the situation to which the quantities on the left hand sides of the equations refer. E.g., the quantity defined by (21) can be regarded as the answer to the question,

> The man ob has been assigned the label λ=tall by the subject.
>
> What is the probability that his height is u^{ex}=185 cm? (26)

If this question is put *to the subject*, then she will use her own estimate of $P(u^{ex})$ for the population of objects to which she believes that the experiment refers. If the question is put *to the experimenter*, then she can use *her* information concerning the $P(u^{ex})$ function. In our example, this may be the distribution over height of all men as published by some official agency. A better approximation to $P(u^{ex}|\lambda)$ is obtained by the experimenter if she uses for $P(u^{ex})$ the relative frequencies of u^{ex} in the particular sample *OB* presented to the subject. If she uses eqn (21) to give her answer to (26), then her estimate of $P(u^{ex}|\lambda)$ is based on the subject's estimate of $P(\lambda|u^{ex})$ (namely $\mu_\lambda(u^{ex})$), and on her own findings concerning $P(u^{ex})$. We may even have the case in which the experimenter chooses a biased sample from the population of objects to which the subject believes that the experiment refers. E.g., a random sample from the set of all policemen (assuming that the average height of policemen is bigger than that of the population as a whole). The experimenter should then use the $P(u^{ex})$ distribution for policemen (or for the particular sample of policemen) in (21) to give her answer to (26).

In the last case, the experimenter's estimate of $P(u^{ex}|\lambda)$ is based on information from mixed sources. Namely on the subject's estimate of $P(\lambda|u^{ex}) = \mu_\lambda(u^{ex})$ for the object to which the question (26) refers; and on her own knowledge concerning the $P(u^{ex})$ distribution. If the subject does not know that the sample used by the experimenter is biased, then she will use her usual quantization intervals for the elements of the label set $\Lambda = \{$small man, medium man, tall man$\}$ to which she refers her membership values. If she knows that she is presented with a biased sample of men, then she may adjust her quantization intervals for the elements of Λ , and displace them towards higher values, even though the question (26) does not refer to policemen. This will again result in a displacement of her membership curves towards higher u^{ex} values. Her $\mu_\lambda(u^{ex})$ values will therefore be difficult to interpret correctly.

We conclude that there exists more than one prior distribution $P(u^{ex})$ which results in mathematically consistent values for $P(u^{ex}|\lambda)$, $P(\lambda)$ and $P^{RR}(\lambda|\lambda)$. The $P(u^{ex})$ distribution which is the correct one from a semantic point of view depends on the situation to which the three quantities refer.

5. More Consequences

In the following we enumerate consequences of the TEE model which make use of the formulas of sect. 4, and thus of the prior distribution $P(u^{ex})$. These consequences concern the resolution of former difficulties, the interpretation of previously postulated formulas and the concepts which they describe, and the necessary qualifications or modifications in connection with other previous concepts or formulas. Most of these items have been listed under difficulties 1, 6, 15 and 17 in Hisdal (1986a).

Consequences 9 and 10 treat the resolution of 'the possibility certainty paradox' and of 'the only man on earth (who is 255 cm tall) paradox'. Both consequences are connected with the differentiation in the TEE model between probabilities/possibilities of $\lambda|u^{ex}$ versus those of $u^{ex}|\lambda$. The lack of differentiation between these distributions in the 'traditional' theory of possibility has been discussed previously (Hisdal 1986a, eqns(4), (5)).

Consequences 11, 12 concern the precisation and defuzzification of the meaning of Zadeh's expressions for the probability of a fuzzy event and for the degree of consistency γ of a probability with a possibility distribution. Finally consequence 13 deals with the entropy formulas which are valid for fuzzy sets. In this case Zadeh's eqn (28) below is replaced by the new equation (35).

There is a second concept whose formula is replaced by a new one in the TEE model, namely Zadeh's 'possibility measure' (Zadeh, 1978a). We have already discussed in Hisdal (1986a, difficulty 17b) that the only meaningful interpretation of this concept seems to be 'the poss/prob that x is b, given that x is a' where a and b are elements of the same or of different label sets. In the TEE model we therefore interpret this measure as $P(b|a)$, the 'autological probability' (or its estimate by the subject) that a given object is assigned the label $b \in \Lambda 2$ in an LB or YN experiment, given that it has been assigned the label $a \in \Lambda 1$ in a previous such experiment. The TEE model formulas for $P(b|a)$ are given in Hisdal (1986a, eqns (16)-(19)) for the SIM and the RR case. They are quite different from Zadeh's max-min formula (eqn (15) in Hisdal 1986a) which makes no use of the prior distribution. This is in contrast to Zadeh's formulas for the negation, the probability of a fuzzy event and the possibility/probability consistency whose meaning is defuzzified in the TEE model, the formulas being left unchanged.

Consequence 9. The resolution of the possibility-certainty difficulty (difficulty 1 in Hisdal, 1986a). As an illustration of this difficulty, consider the case of a grade of membership value

$$\mu_{tall}(u^{ex} = 195\,\text{cm}) = \pi_{tall}(u^{ex} = 195\,\text{cm}) = 1\ . \tag{27}$$

In spite of the value of 1 for this possibility, it is not interpreted as meaning that the outcome u^{ex} =195 cm is a certainty for an object which has been assigned the grade of membership 1 in 'tall'. The theory of possibility has thus no means of distinguishing the important case of a certainty from a mere possibility.

In the TEE model, the difficulty is resolved because the possibility or membership value in (27) is interpreted as the subject's estimate of $P(\text{tall}|u^{ex}) = 1$; i.e., as her estimate that the label 'tall' will *always* be assigned to this object in an LB experiment or in a YN experiment (with λ_{spec} =tall). In contrast, the probability that an object which has been assigned a grade of membership $\mu_{tall} = 1$ has a given height value u^{ex} is computed from eqns (21), (22). It follows from these equations that this probability is always smaller than 1, even though $\mu_{tall}(u^{ex})$ =1, unless $\mu_{tall}(u^{ex})$ or $P(u^{ex})$ are 0 for all other values of u^{ex} .

Consequence 10. The resolution of 'the only man on earth difficulty'. This difficulty concerns a man who is 255 cm tall, and whose grade of membership in 'tall man' is consequently equal to 1 (according to some, and probably all, subjects, assuming that the MU experiment refers to a YN situation; or to an LB situation with the reference label set (1), see Hisdal 1988a, sect. 1 and defs. 4, 5, 15). However, he is the only man on earth with this big height value. Consequently the event that a man who is labeled 'tall' has the height 255 ± 5 cm is an extremely rare one. In spite of this, the grade of membership $\mu_{tall}(u^{ex} = 255\,\text{cm})$ is equal to 1. This value seems to be meaningless in view of the extreme rarity of the event u^{ex} =255 cm for a man who has been labeled 'tall'.

Again, in the TEE model, the grade of membership value 1 means only that a man of this height is certain to be labeled 'tall' in a YN or LB experiment.

The probability that a man who has been assigned in a MU experiment the membership value 1 in 'tall' has a height of u^{ex} =255 cm is obtained from eqn (21). Since $P(u^{ex} = 255\,\text{cm})$ is very small in our case (about 10^{-9} , assuming a world population of 2 billion), it follows from (21) that the probability that a $\mu_{tall} = 1$ man has a height of 255 cm is of the same order of magnitude.

Fig. 4 shows the complete $P(u^{ex}|\text{tall})$ curve as computed from (21), (22) on the basis of the assumed prior distribution $P(u^{ex})$ shown in the figure and the membership curve (also shown in the figure) computed from figs. 2 and 3.

The difference in shape between the probability and possibility curves of 'Hans the egg-eater' (difficulty 15d in Hisdal 1986a), which Zadeh sets up solely by intuition (Zadeh 1978a, p. 8), can be derived in a similar way from the TEE model (Hisdal 1984a, pp. 12.5-12.8).

Consequence 11. The meaning of the probability of a fuzzy event. The right hand side of eqn (22) is identical with Zadeh's postulated formula for 'the probability of the fuzzy event λ' (Zadeh 1968, eqn (5)) whose meaning is far from clear. Eqns (19) and (22) clarify this fuzziness in meaning and show that the 'fuzzy event' λ is no different from an ordinary event in the theory of probability. The right hand side of (19) and (22) is an expression for the (marginal) probabiliy that a randomly chosen object will be labeled λ in a YN or LB experiment, irrespective of the attribute value of the object. In (22), the value of this probability is based on the subject's estimates $\mu_\lambda(u^{ex})$ of $P(\lambda|u^{ex})$ for objects of different atrribute values u^{ex}.

The expression for the probability of a fuzzy event shows also that the prior distribution $P(u^{ex})$ *has* been used in fuzzy set theory, in spite of all claims that this theory has no connection with the Bayesian approach.

Consequence 12. The meaning of the degree of consistency γ of a probability with a possibility distribution. Zadeh (1978a) defines this quantity as $\gamma = \sum_{i=1}^{I} \pi(u_i)\ p(u_i)$. (see Hisdal (1986a, eqn (20) for more details). The $\pi(u_i)$ are numerically equal to the $\mu(u_i)$ membership values according to Zadeh, in our notation to $\mu_\lambda(u^{ex})$. A reasonable interpretation of Zadeh's probability distribution $p(u_i)$ is our $P(u^{ex}|\lambda)$ (although another probability distribution is also relevant in connection with Zadeh's egg-eater example, see Hisdal 1984a, p. 12.8, fig. 12.1b). It then follows from eqn (23), that γ is equal to the $P^{RR}(\lambda|\lambda)$ autological probability as defined in the sequel to eqn (19). In the SIM case, in which no ReRandomization takes place, we have always that $\gamma^{SIM} = P^{SIM}(\lambda|\lambda) = 1$.

Consequence 13. Entropies for fuzzy sets. Zadeh (1968, eqn (28)) defines the entropy of a fuzzy subset λ_l of the finite set $\{x_1, \ldots, x_i, \ldots, x_I\}$ with respect to the probability distribution $P(X) = \{P(x_1), \ldots, P(x_i), \ldots, P(x_I)\}$ by the equation

$$H^{P(X)}(\lambda_l) = \sum_{i=1}^{I} \mu_{\lambda_l}(x_i)\ P(x_i)\ log\,[1/P(x_i)]\ . \tag{28}$$

Alternatively he calls (28) the entropy of the fuzzy event λ_l with respect to the distribution P(X). We shall see that this quantity is replaced in the TEE model by eqn (35).

In Shannon's theory, the 'entropy of the source X', or the expected uncertainty with respect to the outcome x_i, is measured by the quantity,

$$H(X) = \sum_{i=1}^{I} P(x_i)\ log\,[1/P(x_i)]\ , \qquad \text{where} \qquad \sum_{i=1}^{I} P(x_i) = 1\ . \tag{29}$$

When two attributes X, Y are connected with each outcome, then we have also the uncertainty with respect to the outcome of X when the outcome y_j of Y is specified and known to the observer,

$$H(X|y_j) = \sum_{i=1}^{I} P(x_i|y_j)\ log\,[1/P(x_i|y_j)]\ , \quad \text{where} \quad \sum_{i=1}^{I} P(x_i|y_j) = 1\ . \tag{30}$$

The expectation of this quantity over all Y is denoted by $H(X|Y)$, and is called the equivocation of X when Y is known to the observer,

$$H(X|Y) = \sum_{j=1}^{J}\ P(y_j)\ H(X|y_j)\ . \tag{31}$$

The meaningfulness of Zadeh's entropy formula (28) is questionable because many of the mathematical properties of the entropy or equivocation which make these functions significant ones as defining the expected uncertainty before –, or the expected information after – an outcome (see Shannon & Weaver 1964, pp. 48-53, 67; also end of this section), depend on the structure of eqns (29)-(31), and are thus violated in (28). This problem disappears in the TEE model in which the meaning of the different expressions is clearly defined.

Let O be an observer to whom the exact attribute value u^{ex} of the object is known. O's uncertainty with respect to the label λ_l which a subject S will attach to the object under a randomly chosen point of the set of conditions of observation is then found by substituting $P(\lambda_l|u^{ex})$ for $P(x_i)$ in (29), and summing over all elements λ_l of the reference label set Λ,

$$H(\Lambda|u^{ex}) = \sum_{l=1}^{L} P(\lambda_l|u^{ex})\ log\,[1/P(\lambda_l|u^{ex})] = \sum_{l=1}^{L} \mu_{\lambda_l}(u^{ex})\ log\,[1/\mu_{\lambda_l}(u^{ex})] \quad (32)$$

where

$$\sum_{l=1}^{L} P(\lambda_l|u^{ex}) = \sum_{l=1}^{L} \mu_{\lambda_l}(u^{ex}) = 1 \qquad \forall u^{ex}\ . \quad (33)$$

Eqn (32) is the only formula in the present section which makes no implicit or explicit use of the prior distribution $P(u^{ex})$. Note that the summation is performed over the elements of the reference label set, not over u^{ex}.

$H(\Lambda|U^{ex})$, the expected uncertainty (equivocation) with respect to the label λ for objects of any u^{ex} value (this value being known to the observer) is given by the expectation of the right hand side of (32) over the prior distribution $P(u^{ex})$,

$$H(\Lambda|U^{ex}) = \sum_{i=1}^{I} P(u_i^{ex})\ \sum_{l=1}^{L} \mu_{\lambda_l}(u_i^{ex})\ log\,[1/\mu_{\lambda_l}(u_i^{ex})] \quad (34)$$

In contrast, the uncertainty with respect to u^{ex} for objects which have been assigned a specific label λ_l is given by

$$H(U^{ex}|\lambda_l) = \sum_{i=1}^{I} P(u_i^{ex}|\lambda_l)\ log\,[1/P(u_i^{ex}|\lambda_l)]$$
$$= [1/P(\lambda_l)] \sum_{i=1}^{I} P(u_i^{ex})\ \mu_{\lambda}(u_i^{ex})\ log\,P(\lambda_l)/[P(u_i^{ex})\ \mu_{\lambda_l}(u_i^{ex})]\ , \quad (35)$$

where the expression after the last equality sign is obtained from (21).

$H(U^{ex}|\Lambda)$, the *expected* uncertainty (equivocation) with respect to the exact attribute value u^{ex} when the label attached to the object is known to the observer, is given by the expectation of the right hand side of (35) over $P(\lambda_l)$, eqn (22).

Finally $H(\lambda_l)$ is the uncertainty connected with the outcome of the label λ_l for objects whose u^{ex} value is unspecified and unknown to the observer. This is given by

$$H(\Lambda) = \sum_{l=1}^{L} P(\lambda_l)\ log\,[1/P(\lambda_l)]$$
$$= \sum_{l=1}^{L} log\,[1/\sum_{i=1}^{I} P(u_i^{ex})\ \mu_{\lambda_l}(u_i^{ex})] \sum_{i=1}^{I} P(u_i^{ex})\ \mu_{\lambda_l}(u_i^{ex})\ , \quad (36)$$

where the last expression in (36) is obtained from (22).

The $H(\Lambda)$ and $H(\Lambda|\cdot)$ entropies refer to the uncertainties of an external observer concerning the subject's choice of label $\lambda_l \in \Lambda$. The $H(U^{ex})$ or $H(U^{ex}|\cdot)$ entropies are the external observer's uncertainties concerning the experimenter's result of the measurement of u^{ex}, the exact attribute value of the object. When grades of membership are used in the formulas, then the values of the uncertainties are based on the subject's estimate of the probability of assignment of the label λ_l as elicited in a MU experiment.

Equations (32)-(36) refer to an LB or LB-MU experiment. If we want them to refer to a YN or YN-MU experiment, then Λ must be replaced in these equations by the pair-set $\{Y\text{-}\lambda,\ N\text{-}\lambda\}$. The entropies of eqns (32), (34), (36) will then have a value that is smaller than or equal to $\log 2$. For LB reference these entropies are smaller than or equal to $\log L$, where L is the number of elements in Λ. The $H(U^{ex})$, $H(U^{ex}|\lambda_l)$ and $H(U^{ex}|\Lambda)$ entropies are smaller than or equal to $\log I$, where I is the number of quantization points in U^{ex}. Furthermore it follows from a theorem of information theory (Shannon & Weaver 1962 p. 52) that $H(\Lambda|u^{ex})$ and $H(\Lambda|U^{ex})$ are always smaller than or equal to $H(\Lambda)$; and $H(U^{ex}|\lambda_l)$ and $H(U^{ex}|\Lambda)$ are smaller than or equal to $H(U^{ex})$.

In conclusion we note that eqns (32) and (35) are of particular interest. In eqn (32), the probabilities in Shannon's basic entropy equation (29) have been replaced by grades of membership. Such a replacement has previously been suggested by De Luca and Termini (1972, eqn 3. This equation, as a measure of the entropy of a fuzzy set, is then replaced by their somewhat more complicated eqn (9) which is based on eqn (3)). There is, however, a very important difference between De Luca and Termini's eqn (3) and our eqn (32). While the summation in (32) is over the elements λ_l of the reference label set Λ, the summation in De Luca and Termini's equation is over all $u^{ex} \in U^{ex}$. The meaning of the sum over $\mu\ log(1/\mu)$ in (32) is clearly defined in the TEE model. It is the observer's uncertainty with respect to the label assigned by the subject to objects of a given value of u^{ex}.

The observer's uncertainty with respect to u^{ex} 'for a given fuzzy set λ_l', i.e. for those objects to which the subject has assigned the label λ_l, is given by the TEE model equation (35) which replaces Zadeh's postulated eqn (28) (in which x_i should be replaced by u_i^{ex} for purposes of comparison).

6. Conclusion

It has always been implicitly assumed in fuzzy set theory that there exists a connection between the fact that an object is labeled λ (e.g., λ =young) in a natural language discourse on the one hand; and the assignment to the object of a high grade of membership in λ on the other. In the TEE model for grades of membership, this connection is laid down on a firmer, and more quantitative basis through the LB,YN-MU assumption of equivalence, def. 2. According to this assumption, the memberhsip value which a subject assigns to an object in the class λ is her estimate of the probability that this object would be labeled λ in an LB or YN situation in the presence of fuzziness # 1, 2 or 3.

Using this interpretation of grades of membership, we have then shown that the fuzzy set membership function for a label λ is a rounded, or fuzzified version of the nonfuzzy threshold curve used in an LB or YN situation. Furthermore we have shown that the TEE model clarifies the meaning of previously-used concepts of fuzzy set theory, and lets us derive formulas which were previously postulated. The most important clarification in meaning concerns that of the membership concept itself.

Thus the TEE model, in contrast to the usual philosophies of many-valued logics and fuzzy set theory, does not *modify* traditional two-valued logic by smearing out the 0 and 1 truth values of the latter over the whole [0,1] interval, without explaining how this smearing-out process is accomplished. Instead, it leaves the exact logic of YN and LB experiments intact and uses it as a building stone for explaining and deriving the results of MU experiments which make use of intermediate truth values; attributing such values to the subject's estimate of the effects of fuzziness # 1, 2 or 3. This reconciliation between two-valued logic with its law of the excluded middle on the one hand, and the graded membership concept on the other, has been discussed in more detail elsewhere. (Hisdal 1985; the law of the excluded middle is discussed in sect. 5 of that paper.) Two-valued logic can thus be looked upon as a metalanguage used for the definition of the higher-order many-valued logic of the TEE model; both of these languages for the processing of logical information being parts of our basic metalanguage, namely natural language.

Such a stepwise model, in which fuzzy logic is elevated to a higher position than the two-valued logic from which it is constructed, does indeed seem to be a reasonable explanation of the logical aspect of human thinking when we remember 1) that small children insist on precise definitions and information. 2) that the processing of imprecise information is much more complicated and memory-consuming than that of precise information. Zadeh's seemingly convincing principle of incompatibility between precision and significance (Zadeh 1973, p. 28) is true only when we identify precision with oversimplification. Such an oversimplification is illustrated by Zadeh's example of the definition of 'recession' as a condition which obtains when the gross national product declines in two successive quarters (Zadeh 1976, p. 251). The lack of significance of this definition should be a signal to the economist that the single attribute 'ratio between gnp in successive quarters' is not sufficient for defining a recession. If we succeed in identifying all the attributes necessary for the definition of a recession, as well as the situations under which these attributes are elicited, then we have the tool for a precise definition which does not lack in significance. The significance being due to the correct identification of the high-dimensional attribute universe, not to the use of fuzziness as a basic concept. Fuzziness is introduced when we do not, or cannot, take the values of some of the attributes into account; this being a typical case of fuzziness #2 (Hisdal 1986b). In such a situation the use of an intermediate membership value is indeed a very important tool for the best possible processing of the available information.

An objection to the TEE model on the ground that it sometimes makes use of a prior probability distribution cannot be maintained when we consider that prior distributions *have* previously been used for defining the probability of a fuzzy event and the entropy of a fuzzy set (consequence 11 and eqn (28)). Likewise the mystical 'particularizing distribution' used by Zadeh (see Zadeh 1978b, p. 407; or Hisdal 1980, def. 3.1) is a prior distribution.

The most important previous formulas which are not generally confirmed in the TEE model are the max and min formulas for OR and AND respectively. These are precisely the formulas for which alternatives have been suggested throughout the years, both by Zadeh and by many other researchers. (See, e.g., Zadeh, 1975 p. 34 eqns (A49), (A50); Zadeh, 1973, footnote p. 31; Zadeh, 1978b p. 425; Dubois & Prade, 1980, p. 16; Yager 1978.)

Because the TEE model does not accept the general validity of the max and min operations, it cannot hope for support from that part of the purely mathematically-minded group of fuzzy set workers who consider these operations to be the cornerstone of fuzzy set theory.

However, the TEE model should have a clear message to those who are interested in establishing a theory which can explain the use of-, and the reasoning with-, linguistic labels, negation, connectives and modification by 'VERY' in everyday language and discourse. A logic which is in accord with this reasoning has the biggest chance of being useful for extensions and applications to more complicated cases because it evolves out of the basic metalanguage used for the definition of every logical system.

Although natural languages differ widely as to their syntax and their use of sounds, it seems that the basic logical operations are common to most natural languages, and are therefore, probably, connected with the structure of our brains. Higher order logical systems cannot escape the necessity of coping with this structure. As long as we cannot explain the means of reasoning in our basic metalanguage, we cannot hope to construct completely consistent higher-level logical systems without paradoxes; let alone automated reasoning systems whose output agrees in *all* limiting special cases with the output expected by human beings according to their most basic logic.

PRINTING ERROR

The second paragraph of section 1 of the first paper of this series (Hisdal, 1986a), which was correctly printed in the proof of the paper, lost some of its lines in the final edition of the journal, while others were interchanged. The correct formulation of this paragraph is:

There are, maybe, those who claim that logical systems should always be independent of the use of logic in everyday discourse. But we should remember, that no matter how many successive metalanguages we use to describe our logical system, the final metalanguage must always be natural language. Thus the logic of natural language stands in a unique position compared with all other systems of logic. If we deny the functioning of the logic of natural language, then we also deny the possibility of the description of any other logical system. Furthermore, fuzzy set theory concerns, according to Zadeh (1973, 1976) the use and logic of fuzzy expressions in natural language.

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Appendix A1. An Alternative LB,YN-MU Assumption B

It seems to me that the first and second assumptions of the TEE model, see sect. 1.1, describe the only possible procedures that a subject can use when she wants to give meaingful answers to a semantic experiment.

This is not quite true of the third or LB,YN-MU assumption of def. 2 in sect. 2.2. There exists an alternative meaningful procedure for specifying numerical membership values. This procedure is described in def. 4 below. Here we will call the LB,YN-MU assumption of def. 2 'assumption A' and that of def. 4 'assumption B'.

There are two important differences between the predictions of assumptions A and B. The first concerns the membership values specified in an exact MU experiment in which the subject is acquainted with u^{ex}, the exact attribute value of the object. Assumption B of this appendix predicts that she will always specify nonfuzzy membership values (of either '0' or '1') in such an experiment. This is in contrast to assumption A according to which the subject can specify intermediate membership values also in an exact experiment.

The second difference between the two LB,YN-MU assumptions concerns the use of the prior distribution $P(u^{ex})$. According to assumption A, the subject makes no use of this distribution when she assigns her membership values in any MU experiment. In contrast, she needs to estimate this distribution when she gives her answers according to the procedure of assumption B in a nonexact MU experiment.

Traditional fuzzy set theory has always assumed that membership values are independent of the prior distribution. It has also assumed that there exists a unique $\mu_\lambda(u^{ex})$ function (for a given subject and a given context dependence of λ) whose range is the whole [0,1] interval*.

These assumptions of traditional fuzzy set theory, which probably also agree with experimental results, can only be reconciled with the (LB,YN-MU)-A assumption. However, since assumption B is just as consistent from a logical

* A unique $\mu_\lambda(u^{ex})$ function can exist only for an exact MU experiment. A nonexact MU experiment will always result in the specification of variable membership values for objects of a given u^{ex}. (See Norwich & Turksen, 1982; also Hisdal, 1986b sect. 3 and p. 134.)

point of view, we present it here. Assumptions A and B correspond roughly to the use of likelihoods versus that of a posteriori probabilities which, around 1916, gave rise to the big controversy between the two great statisticians Fisher and Pearson (see Fisher-Box, 1978, e.g. pp. 68, 70, 79, 89).

Before we present assumption B, we note that our previous statement that a subject probably makes no use of a prior distribution when she gives her answer in a MU experiment concerns only the conversion from LB or YN experiments to MU experiments in the subject's mind as given by eqn (5). The choice of the quantization interval Δu_λ for λ is probably strongly influenced by her estimate, $P^{est}(u^{ex})$, of the prior distribution. The description of a person as being 'small' or 'tall' implies that the subject's estimate of the person's height deviates from the estimated average height value; and the latter depends on $P^{est}(u^{ex})$.

Definition 4 of the (LB,YN-MU)-B assumption or assumption 3-B of the TEE model. According to this assumption, a subject who performs a MU experiment estimates both the attribute value of the object and the $P(u|u^{ex})$ error function for the particular condition of observation under which she observes the object in the particular experiment. She then specifies a μ_λ value for the object which is equal to her estimate of the probability that u^{ex}, the exact attribute value of the object, falls into her quantization interval Δu_λ for λ,

$$\mu_\lambda(\text{object}) = \mu_\lambda(u) = P^{est}(u^{ex} \in \Delta u_\lambda \mid u) \ . \qquad (A1)$$

When the MU experiment is an exact one, then the subject's estimate u of the object's attribute value is equal to u^{ex}. $\mu_\lambda(\text{object}) = \mu_\lambda(u^{ex})$ of eqn (A1) is then equal to the subject's nonfuzzy $P(\lambda|u^{ex})$ function which she uses in an LB or YN experiment (see the nonfuzzy threshold curves in fig. 2).

For a nonexact MU experiment, the procedure of assumption B results in a *distribution* of μ_λ values for objects of a given u^{ex} just as for the LB,YN-MU assumption A (see last footnote here and eqn (A9) in Hisdal 1986b). The expectation value of $\mu_\lambda(u^{ex})$ is then given by,

$$\text{Exp}\ \{\mu_\lambda^{nexcond}(u^{ex})\} = \sum_u P(u|u^{ex})\ \mu_\lambda(u) \ , \qquad (A2)$$

where $P(u|u^{ex})$ is the subject's real error curve under the given set of conditions of observation. We show below that $\mu_\lambda(u)$ on the right hand side of eqn (A2) is given by,

$$\mu_\lambda(u) = [1/P^{est}(u)] \sum_{u^{ex\prime} \in \Delta u_\lambda} P^{est}(u^{ex\prime})\ P^{est}(u|u^{ex\prime}) \ , \qquad (A3)$$

where

$$P^{est}(u) = \sum_{u^{ex\prime}} P^{est}(u^{ex\prime})\ P^{est}(u|u^{ex\prime})\ . \tag{A4}$$

In these equations, $u^{ex\prime}$ denotes the various possible values which u^{ex} may assume for the given object according to the subject's estimate. Furthermore, all probability distributions are estimates by the subject. Note that $P^{est}(u|u^{ex\prime})$ is, in contrast to the case of assumption A, not a constant function referring to the set of everyday conditions of observation. It is the subject's estimate of the error function under the given condition of observation (i.e. the given point of the set of conditions of observation, see Hisdal 1986b, end of sect. 1) under which the particular object to which she assigns a MU value is observed.

end def. 4

To prove (A3), (A4), let u^{ex} be the attribute value of the object as measured by the experimenter, and let u be the subject's estimate of this value. We then have from the law of compound probabilities that the following relation holds between the real probability distributions pertaining to a statistical experiment whose outcomes are the values of u^{ex} and u for a given object,

$$P(u^{ex}, u) = P(u^{ex})\ P(u|u^{ex}) = P(u)\ P(u^{ex}|u), \tag{A5}$$

and therefore

$$P(u^{ex}|u) = [1/P(u)]\ P(u^{ex})\ P(u|u^{ex})\ , \tag{A6}$$

where

$$P(u) = \sum_{u^{ex}} P(u^{ex},\ u) = \sum_{u^{ex}} P(u^{ex})\ P(u|u^{ex}). \tag{A7}$$

Eqn (A1) tells us that the $\mu_\lambda(u)$ value specified by the subject depends on her estimate of the $P(u^{ex}|u)$ distribution pertaining to the given condition of observation. To estimate this distribution, the ideal (LB,YN-MU)-B subject substitutes her estimates of $P(u^{ex})$ and $P(u|u^{ex})$ into the right hand sides of (A6) and (A7). Substituting the resulting eqn (A6) into (A1), we then obtain (A3); while (A4) follows from (A7). q.e.d.

In summary, in a nonexact MU experiment, the expectation of the $\mu_\lambda(u^{ex})$ membership curve is given, according to assumption B, by the right hand side of (A2), $\mu_\lambda(u)$ being given by (A3), (A4). The membership curve elicited in an exact MU experiment is, according to assumption B, identical with the nonfuzzy threshold curve elicited from the same subject in an exact LB or YN experiment. Norwich and Turksen's experiment, section 2.3, thus comes very near to falsifying assumption B.

Appendix A2. The TEE Model and Bandler and Kohout's Checklist Paradigm

We show in this appendix that the TEE model can be derived from Bandler and Kohout's (1985) checklist paradigm, provided that we add to the latter the asssumption that grades of membership specified by a subject in a MU experiment are estimates by the subject of the truth values elicited according to this paradigm in one of the three experimental situations desribed below.

BK's (Bandler and Kohout's) paper deals mainly with the connectives and the implication. Their checklist is then a two-dimensional table. In Hisdal (1988b), we come back to the connection between the checklist paradigm and the TEE model in this case.

Here we make use only of BK's simplest checklist consisting of a one-dimensional table with n originally empty entries (see BK sect. 1, first two paragraphs). The table is used in an experimental situation in order to find the degree of truth of a statement A which summarizes a sequence of n detailed assertions $A_1, \ldots, A_n$. The subject is asked to give a Y or N answer to each of these n assertions. The degree of truth of A is then defined as $a = n_Y/n$, where n_Y is the total number of Y answers. At the end of their sect. 2, the authors also use the symbol μ for n_Y/n instead of a.

To apply this checklist to the simple case of finding the degree of truth of the statement A that an object of attribute value u^{ex} is λ (e.g., that a man of height 175 cm is tall), we let the n detailed assertions $A_1, \ldots, A_n$ be identical, namely:

$$\text{This object is } \lambda \ . \tag{A8}$$

For each assertion, the subject is shown an object with the same exact attribute value $u^{ex} = u_i^{ex}$. However, each of the n assertions is presented in a new situation. The type of variation in situation depends on the type of fuzziness to which the $\mu_\lambda(u^{ex})$ value is to refer (see Hisdal 1986b, fuzziness # 1a, 2a, 3a).

In connection with fuzziness #1a, the object is observed under a new condition of observation for each statement (more precisely, under a new, randomly chosen, point of the set of conditions of observation.)

Fuzziness # 2a deals with the case of a two-attributional concept λ (e.g., 'slimness' which refers to the universe $U^{ex} \times W^{ex}$, where u^{ex} =height of object and W^{ex} =weight of object) in the case when the subject is acquainted with (or can estimate) the value of one of the attributes only (e.g., the height u^{ex}). For each assertion, the subject is presented with a new object chosen at random from all objects whose first attribute has the value u_i^{ex}. The value of the second attribute is randomly distributed and varies from assertion to assertion. (The example of λ =slim is not a good one in connection with fuzziness # 2a because the probability of being slim or stout is largely independent of the person's height. For a better illustration, see the medical diagnosis example in Hisdal 1986b, p. 121.)

For fuzziness # 3a, the subject who gives the YN answer varies from assertion to assertion, being chosen at random from the population of subjects.

For each of these three types of fuzziness, the experimenter then defines the truth value of $A = A_1, \ldots, A_n$ as being equal to n_Y/n, where n_Y is the total number of Y answers.

To obtain the TEE model, we only add the assumption that the membership value specified by a subject in an exact MU experiment (with YN reference, see Hisdal 1988a, def. 5) is an estimate of n_Y/n in connection with fuzziness # 1a, 2a, or 3a.

We have thus complete correspondence between the TEE model (with YN reference of the membership values) and the above application of the BK checklist paradigm. Partial truth or grade-of-membership values are considered to be estimates by the subject of the average of the Y (1) and N (0) answers.

Two additional, extremely important results which are common to the checklist paradigm and the TEE model are 1) The unique identification of grades of membership with distributions of $Y\text{-}\lambda|u^{ex}$, not of $u^{ex}|Y\text{-}\lambda$. And 2) The summing-up-to-1 of the truth or membership value of $Y\text{-}\lambda|u^{ex}$ and $N\text{-}\lambda|u^{ex}$ for all u^{ex}. These two results make it quite clear that the fact, that the sum over u^{ex} of the ordinates of a discrete membership function is usually bigger than 1, is not a valid argument for the need of a non-additive measure theory.

Appendix A3. The Resolution of the Complementation Paradox

We show in this appendix that the complementation paradox of fuzzy set theory disappears when the traditional *postulated* max operator for the ORA (inclusive OR) connective is replaced by the *derived* TEE model operators for this connective.

Let $\mu_a(u^{ex})$ be the membership function of the fuzzy subset a of the attribute universe U^{ex}. And let $NOT\ a$ be the complement of a. It is defined as a fuzzy subset whose membership function is that of the negation of a as given by eqn (9) (see Zadeh, 1973, eqn 2.26)).

The max-min fuzzy set theory postulates the pointwise max operator for the union of two fuzzy subsets (see Zadeh 1973, eqn (2.27)). We have then the paradox that the union of a fuzzy subset a of U^{ex} and of its complement $NOT\ a$ is, in general, not equal to the attribute universe U^{ex}. In the sequel we show that the use of either of the two TEE model operators for ORA (corresponding to SIM and RR situations respectively) results in the attribute universe for the union of a fuzzy subset and its complement, see eqn (A15).

We shall here use the implied definition of traditional nonfuzzy and fuzzy set theory according to which the union of two fuzzy subsets a, b, is the fuzzy subset whose membership function is $\mu_\lambda(u^{ex})$, where $\lambda = a$ ORA b.

The label

$$\lambda = a\ ORA\ b \tag{A9}$$

(with YN reference for the composite label λ as a whole) applies, according to the TEE model, to every object which is such that when two successive LB or YN experiments, exp1 and exp 2, concerning the noncomposite labels a and b respectively are performed on it, then the following outcomes contribute to $a\ ORA\ b$, assuming that exp1 and exp2 refer to the same label set,

$$\begin{aligned} &a \text{ in exp1 and (either } a \text{ or } b) \text{ in exp2]} \qquad \text{or} \\ &[b \text{ in exp1 and (either } a \text{ or } b) \text{ in exp2].} \end{aligned} \tag{A10}$$

In our case the two experiments refer to the label set

$$\Lambda = \Lambda 1 = \Lambda 2 = \{a, b\}, \qquad \text{where} \quad b = NOT\ a\ . \tag{A11}$$

The quantization intervals for the two labels are such that

$$\Delta u_a \cup \Delta u_b = U\ . \tag{A12}$$

The reason why eqn (A12) must hold is that the subject *must* answer either Y or N in each of the two noncomposite experiments, assuming that these are of the YN type. If they are LB experiments concerning the label set (A11), we get again eqn (A12) according to def. A1 in appendix A1 of Hisdal (1988a).

Two formulas for the ORA connective apply to our case, depending on the situation to which the subject (who assigns a membership value concerning the label $a\ ORA\ b$ in a YN-MU experiment, see Hisdal 1988a, def. 5) refers. For a SIM reference, the subject refers to the case in which the noncomposite LB or YN exp1 and exp2 are performed SIMultaneously on each object. Consequently u, the estimated attribute value of the object, is the same for exp1 and exp2. The formula for the membership function of λ, eqn (A9), is then given by eqns (A13), (A14) below (eqns (10) and (12) in Hisdal (1988c)),*

$$\mu_\lambda(u^{ex}) = \sum_{u \in \Delta u_\lambda} P^{est}(u|u^{ex}) \ , \tag{A13}$$

where Δu_λ is given by,

$$\Delta u_\lambda = \Delta u_a \cup \Delta u_b = U \ . \tag{A14}$$

The last equality sign in (A14) is due to (A11), (A12).

Substituting (A14) into (A13) we find the desired result

$$\mu_{a\ OR\ NOT\ a}(u^{ex}) = 1 \qquad\qquad \forall u^{ex} \ , \tag{A15}$$

because the sum of the probabilities of all possible estimated attribute values u (for a given u^{ex}) must be equal to 1.

A composite label can also refer to an RR situation in which each object is ReRandomized with respect to conditions of observation between exp1 concerning the label a, and exp2 concerning the label b. This statement holds for fuzziness # 1a. For fuzziness # 3a, an RR reference situation means that the subject who performs the MU experiment concerning the composite label refers to the case that two, generally different, randomly chosen subjects assign the labels in exp1 and exp2 respectively. For fuzziness #1a we then have the following formula for ORA (eqn (15) in Hisdal (1988c)),

$$\mu_{a\ ORA\ b}(u^{ex}) = 1 - \{1 - [\mu_a(u^{ex}) + \mu_b(u^{ex})]\}^2 \ . \tag{A16}$$

* In Hisdal (1988c, sect. 2.9), the formulas for the connectives are presented without proof. The proof can be found in Hisdal (1984a), sect. 10 and in Hisdal (1988b).

Substituting the 'one-minus' formula for the negation, $\mu_a(u^{ex}) + \mu_{NOT\ a}(u^{ex}) = 1 \quad \forall u^{ex}$ (see consequence 2 in sect. 3) into (A16), we find that eqn (A15) holds also for RR reference of the membership values of the composite label.

This then is the resolution in the TEE model of the complementation paradox of the max-min fuzzy set theory.

We remark that the formal proofs of this appendix are, strictly speaking, superfluous, except that they demonstrate that the TEE model formulas are consistent. Eqn (A15) must always hold, irrespective of SIM or RR reference of the composite label, because 1) The subject interprets the grade of membership of '*a ORA NOT a*' as her estimate that an object will be labeled '*a*' in one of the two noncomposite YN or LB experiments, ORA that the object will be labeled '*NOT a*' in the other; 2) One of these two labels must necessarily be assigned in each of the two noncomposite experiments; and finally because 3) Every object has a unique exact attribute value u^{ex}.

The complementation paradox has been discussed previously under difficulty 9 in Hisdal (1986a). We said there that the TEE model resolves this paradox by recognizing that a fuzzy subset of the attribute universe is not a collection of elements of U^{ex}, but a distribution over U^{ex}. In contrast, the present appendix shows that the complementation paradox is simply due to the use of a wrong operator, namely the max operator, for the inclusive OR. The appendix therefore represents a modification of our previous statement under difficulty 9. The question of whether we should use the name 'fuzzy subset *of the attribute universe* U^{ex}' for the $\mu_\lambda(u^{ex})$ membership function will be discussed in part 1.5 of this series.

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Δu_λ	$= \{u_{\lambda l}, \ldots, u_{\lambda u}\} =$ quantization-interval for the label λ. Nonfuzzy interval in U for which the subject assigns the label λ in an LB (labeling) or YN (yes-no) experiment (see Hisdal 1988a, definition 11).
est	superscript for the estimate by the subject of a probability distribution relating to his fuzziness #1a, 2a or 3a (Hisdal 1986b).
est-nexcond	superscript for a quantity elicited under real conditions of observation which are identical with the estimated ones assumed by the subject in connection with his fuzziness #1a.
Exact	conditions or experiment. A semantic experiment in which the subject measures (or is told) the exact attribute value of each object. Consequently $u = u^{ex}$ for an exact experiment.
excond	superscript for a quantity elicited under exact conditions of observation for which $u = u^{ex}$.
Fuzziness #1a	is due to the subject's awareness of the possibility of errors of estimation of the attribute value (see section 2.2).
Fuzziness #3a	is due to the subject's awareness of the possibility of variations between different persons of the thresholds u_λ for the label λ (Hisdal 1986b).
λ	a label; e.g. 'tall', or 'VERY tall', or 'tall OR medium'. The same symbol is used to denote the corresponding fuzzy class. In fuzzy set theory it is usual to identify this concept with its membership function $\mu_\lambda(u^{ex})$, and to call it the "fuzzy set λ". We use a lower case letter to denote this concept and its label instead of the more usual A or F of fuzzy set theory, because we need the corresponding upper case letter for a label set.
$\Lambda = \{\lambda_l\}$	a label set; e.g., '$\{$small, medium, tall$\}$' (see Hisdal 1988a, def. 1, item 9).
LB experiment	a Labeling experiment in which a subject assigns a label from a label-set to an object (see Hisdal 1988a, definition 2).
LB,YN-MU	assumption of equivalence of the TEE model. The assumption that the grade of membership value is a means by which a subject takes account of the existence of fuzziness in everyday life. He interprets the μ_λ value which he assigns to an object of attribute value u^{ex} (in a MU experiment performed under exact conditions) as the proportion of objects with that value of u^{ex} which he would label λ in an LB or YN experiment under the conditions of observation to which he refers his fuzziness #1a: $\mu_\lambda^{excond}(u^{ex}) = P^{est-nexcond}(\lambda \mid u^{ex}) = \sum_{u=-\infty}^{\infty} t_\lambda(u)\ P^{est}(u \mid u^{ex})$. For fuzziness #3a, the subject interprets μ_λ as the proportion of subjects who would label the object λ in an LB or YN experiment (Hisdal 1986b).
μ_λ	membership value in class λ assigned by the subject to a given object under given conditions of observation. μ_λ is a unique function of u, but not of u^{ex} according to the TEE model; assuming a given reference label set Λ, and a given reference to either a YN or an LB situation (see Hisdal 1988a, definitions 4, 5).
$\mu_\lambda^{nexcond}(u^{ex})$	Under nonexact conditions, μ_λ is a unique function of u, but not of u^{ex}. We therefore define its expectation over all objects with attribute value u^{ex}, Exp $\{\mu_\lambda^{nexcond}(u^{ex})\} = \sum_{u=-\infty}^{\infty} P(u \mid u^{ex})\ \mu_\lambda^{excond}(u)$. Thus $\mu_\lambda^{excond}(u^{ex})$ is a rounded version of the nonfuzzy threshold curve $t_\lambda(u)$; and Exp $\{\mu_\lambda^{nexcond}(u^{ex})\}$ is a rounded version of $\mu_\lambda^{excond}(u^{ex})$. The rounding-off being performed by a convolution with $P^{est}(u \mid u^{ex})$ and $P(u \mid u^{ex})$ respectively.

Fig. 1a. Notation and Terminology (continued in fig. 1b).

MU experiment	a grade of membership experiment in which the subject is asked to assign a grade of membership value $\mu \in \{0, \ldots, 1\}$ to an object concerning the label λ (see Hisdal 1988a, definitions 4, 5).
$nexcond$	superscript for a quantity elicited under nonexact conditions of obervation. Note that u^{ex} (the exact attribute value of the objects as measured by the $experimenter$) may be an argument such a quantity.
$P(\lambda\|u^{ex})$	labeling probability or likelihood distribution of λ over u^{ex}. Probability that an object with attribute value u^{ex} will be labeled λ in a YN or LB experiment. Superficially stated, it is later identified with $\mu_\lambda(u^{ex})$ elicited in a MU experiment (see LB,YN-MU assumption, sect. 2.2).
$P(u^{ex})$	unqualified or prior probability distribution over u^{ex}; e.g., the distribution over height of the population of objects, unqualified by the label λ.
$P(u\|u^{ex})$	real error curve for a given subject, and a given set of conditions of observation. When $P(x)$, the probability of an error $x = u - u^{ex}$, is independent of u^{ex}, then we talk about a 'u^{ex}-invariant' error curve.
$P^{est}(u\|u^{ex})$	estimated error curve; the subject's estimate of the error curve for the conditions of observation to which he refers his fuzziness #1a.
Semantic	experiment. An LB or YN or MU experiment (Hisdal 1986b, defs. 2-5).
Set of conditions of observation	(see Hisdal 1986b, end of sect. 1); A given set of conditions of observation gives rise to a unique $P(u\|u^{ex})$ error curve for a given subject. For exact conditions of observation the set contains one point only. A set of conditions of observation containing more than one point (anticipated by the subject or real) gives rise to fuzziness #1 (1a or 1b respectively).
$t_\lambda(u)$	threshold curve for λ; a two-valued function of u whose value is 1 inside the quantization interval Δu_λ, and 0 outside this interval. $t_\lambda(u) = P^{excond}(\lambda\|u) = P^{nexcond}(\lambda\|u)$. In contrast, $P^{nexcond}(\lambda\|u^{ex})$ is a fuzzified version of the threshold curve. It is identified with the membership curve elicited from an ideal subject under exact conditions (see sect. 2, defs. 2, 3).
$u_{\lambda l}, u_{\lambda u}, u_\lambda$	$u_{\lambda l}$, $u_{\lambda u}$ are nonfuzzy lower and upper threshold values in U of a given subject for classifying an object as being λ in a YN or LB experiment. For extremal concepts (like 'tall', 'small', 'VERY small') only one of these need to be specified. It can then be denoted by u_λ.
u	estimate of the object's attribute value by the subject.
u^{ex}	exact attribute value of object as measured by the experimenter; e.g. the height in centimeters, measured with a centimeter stick.
U^{ex}, U	the universe in which u^{ex} and u take on values. In all the formulas and figures we assume a quantized universe (although we often leave out a subscript on the quantized 'u' values in order not to complicate the appearance of the formulas). $u = 165$ cm in a figure should be interpreted as $u \in [160, 170)$ cm. Continuous curves are drawn through the computed points for convenience of visualization.
YN (yes-no) experiment	an experiment in which a subject answers 'yes' or 'no' to the question of whether an object is λ (see Hisdal 1988a, def. 3).

Fig. 1b. Notation and Terminology, contnd from fig. 1a.

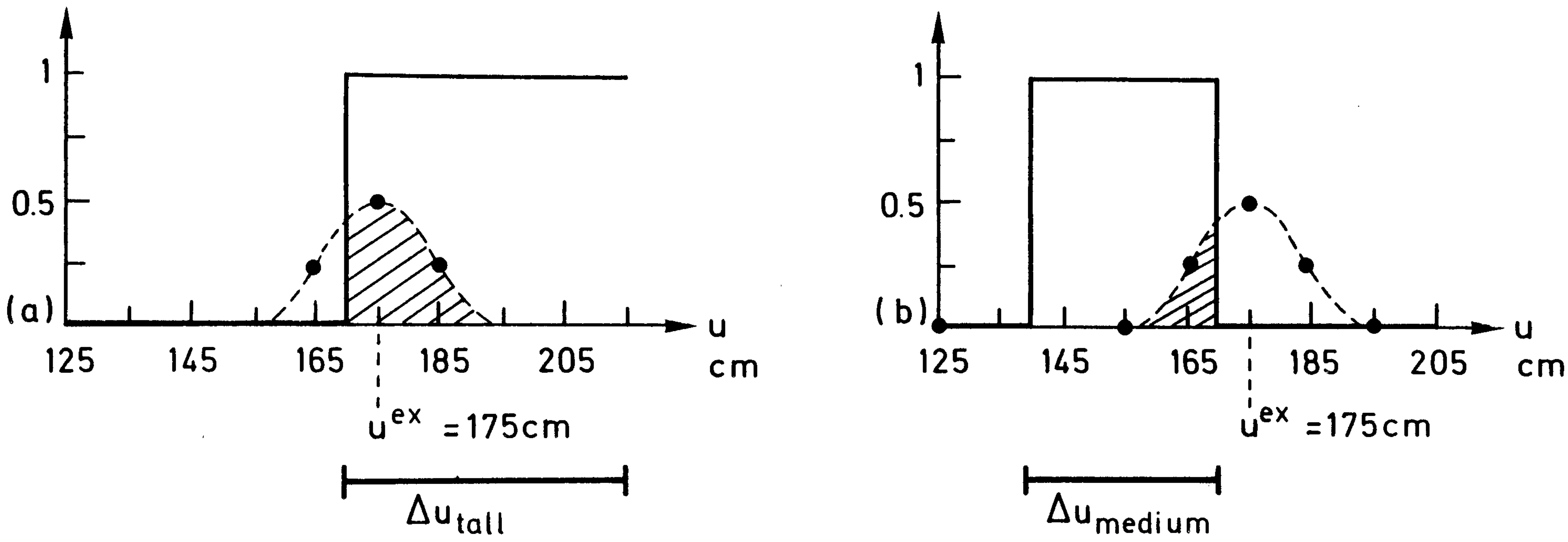

Fig. 2. Derivation in the TEE model of the u^{ex} =175 cm membership value for 'λ =tall' and 'λ =medium' respectively in connection with fuzziness #1a. The membership value is equal to the area (actually sum of ordinates) cut-off by $t_\lambda(u) = P(\lambda|u)$, the nonfuzzy YN or LB threshold curve for λ, from the estimated error curve of fig. 3(a) displaced to $u = u^{ex}$ =175 cm. u is the subject's estimate of the object's attribute value u^{ex}. In contrast to the fuzzy $P(\lambda|u^{ex})$ threshold function of fig. 4 (which is elicited under nonexact conditions of observation), the nonfuzzy $P(\lambda|u)$ threshold function of u (for a given context dependent label λ) is valid for any conditions of observation according to the first and second assumptions of the TEE model, sect. 1.1.

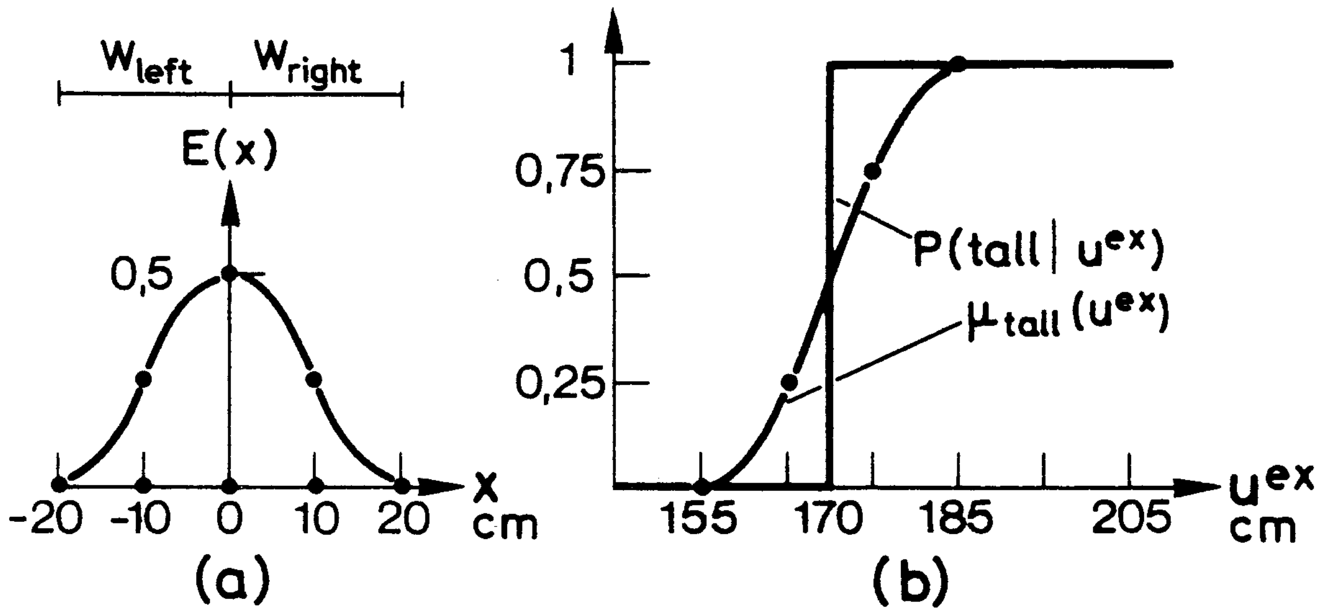

Fig. 3. Fig. 3(b) illustrates the difference, according to the TEE model, between a $P(\text{tall}|u^{ex})$ YN-curve and a $\mu_{tall}(u^{ex})$ membership-curve elicited under the same (namely exact) conditions of observation (see sect. 2.3, theorem 1). It is assumed that the subject's threshold for 'tall' is 170 cm, and that her (u^{ex}-invariant) estimated error curve $E(x) = P^{est}(u - u^{ex}|u^{ex})$ is that of fig. 3(a). The grade of membership curve is computed from the threshold and the error curve according to eqn (3) (see illustration of this equation in fig. 2 for u^{ex} =175 cm). The membership curve is also equal to the $P(\lambda|u^{ex})$ curve which would be elicited from the same subject in a *non*exact YN experiment with a real error curve equal to the one estimated by the subject for everyday conditions of observation in connection with her assignment of the membership values (see LB,YN-MU assumption, sect. 2.2). The difference between a YN and a MU curve elicited under the same conditions of observation has been demonstrated experimentally by Norwich and Turksen.

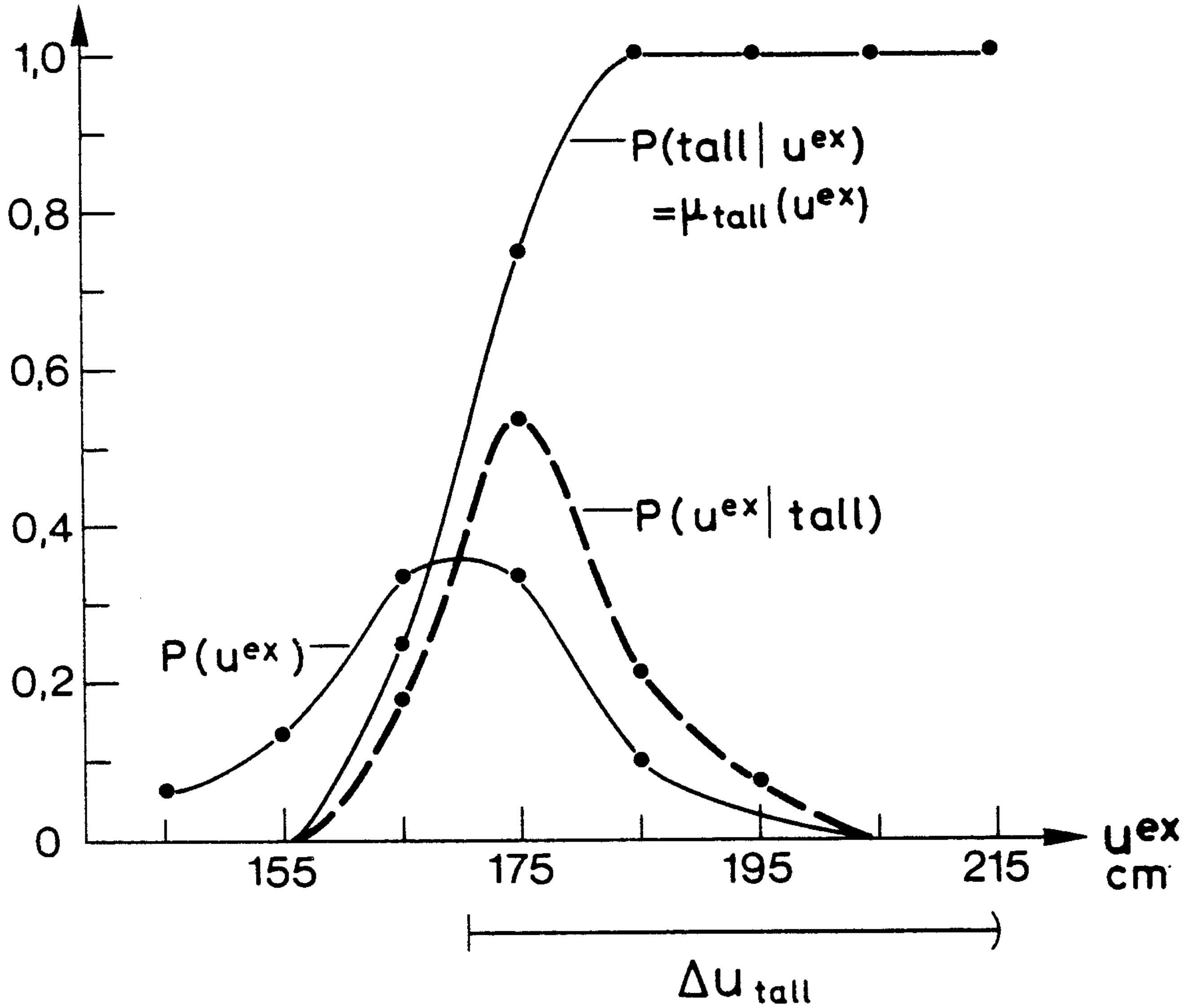

Fig 4. **Likelihoods and grades of membership $P(\lambda|u^{ex}) = \mu_\lambda(u^{ex})$ (elicited in a nonexact and an exact experiment respectively, see eqn (5)) versus λ-qualified probabilities $P(u^{ex}|\lambda)$ (referring to a nonexact YN experiment for the assignment of λ). $P(u^{ex})$ is an assumed unqualified probability distribution (distribution over height of the population to which 'λ=tall' refers). $P(\text{tall} \mid u^{ex})$ is computed from the subject's assumed quantization interval for 'tall' by the method of fig. 2(a) and eqn (3). $P(u^{ex} \mid \text{tall})$ is equal to the normalized product of the other two distributions, see eqns (18), (21). The differentiation between the distribution of $(\text{tall} \mid u^{ex})$ and that of $(u^{ex} \mid \text{tall})$ disposes of 'The Only Man on Earth' difficulty, see consequence 10.**