Infinite-Valued Logic Based on Two-Valued Logic and Probability

Part 1.3. Reference Experiments and Label Sets

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Abstract: The TEE model for grades of membership claims that there is a connection between the fact that an object is labeled λ (e.g., λ =old) in a natural language discourse on the one hand; and the assignment to the object of a high grade of membership in λ on the other. In preparation for a more precise formulation of this assertion we define here 1) LB (labeling) and 2) YN (yes-no) experiments which simulate the natural language situation; and 3) MU experiments which elicit subjective membership values. It is shown that the results of such experiments can be ambiguous unless they are referred to a given label set such as $\Lambda = \{$VERY young, young, middle-aged, old, VERY old$\}$. Furthermore the results of MU experiments can be highly ambiguous unless the subject knows whether they refer to an LB or YN situation. Starting with the simplest case of an exact LB experiment, and working our way up to a nonexact experiment, we present some reasonable assumptions of the TEE model concerning human information processing in connection with the assignment of labels to objects in a natural language situation. Such labels are chosen from a 'nonredundant label set' Λ which partitions the universe U of *estimated* attribute values, resulting in a nonfuzzy step – or square-pulse – shaped function of U for each element of Λ. The softening of this curve to the S- or bell-shaped grade of membership curve elicited in a MU experiment is discussed in the next paper. The fuzzy set 'one-minus' postulate for the negation, and the experimentally observed displacement between the membership curve of a label and its 'VERY' modification, as well as between the negation of a label and its antonym, are *derived* from the simple assumption of subjective, nonfuzzy thresholds of a given label for YN and LB (but not for MU) experiments in the universe of *estimated* attribute values.

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Figures and Contents of
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1. Introduction

This paper is the final preparatory one in the series on the TEE model for grades of membership (Hisdal, 1986a,b). It is followed by the main paper on the interpretation of the membership concept (Hisdal, 1988a).

The present paper has several important goals. One of these is to investigate the situations in which adjective labels are used in everyday discourse, and to lay down formal definitions of experimental situations which correspond to the everyday ones. In addition we define experimental situations in which grade of membership values are elicited. These do not directly correspond to everyday situations because partial membership values are not used in everyday discourse. We shall see, however, that it is important to establish a connection between the formal MU experiment and the everyday situation to which the membership value refers. In some cases different reference situations result in completely different membership curves for the same label. This statement holds not only for the well known effect of the context dependence of the label on the noun, or noun phrase, to which it is implicitly or explicitly attached (e.g. the membership curves of 'old man' versus 'old dog' as functions of the age in years). In addition, the membership curve can depend strongly on whether it refers to an LB (labeling) or to a YN situation.

Consider, e.g., the use of the label 'tall' in an LB situation, in which it is chosen voluntarily; such as in the sentence 'X is tall'. And compare this with a 'Y' (yes) answer to the question 'Is X tall?'. The set of objects to which 'tall' applies in the LB situation is, in general, a subset of the set of objects to which it applies in the second, or YN situation. The reason is that a 'Y' answer in the YN situation is appropriate to objects of very large height which, in the LB situation, would not be labeled 'tall' but rather 'VERY tall'. This seemingly trivial fact influences strongly the shape of the membership curve for 'tall'. The curve referring to the YN situation is S-shaped while that referring to the LB situation is bell-shaped, going to zero for very large height values. Such inconsistencies in shape of the membership curves for 'small' and for 'large' have been observed by Hersh and Caramazza (1976). (See fuzziness #8 in Hisdal (1986b)). In their MU experiments, as well as in the MU experiments of a number of other investigators, it was not made clear to the subject whether the membership values should refer to an LB or a YN situation.

A similar and even more unexpected effect occurs for a label such as 'tall ORA medium' (ORA=inclusive OR). This type of label will be used in a voluntary or LB situation when the subject S estimates the height of the object to lie on the borderline between 'tall' and 'medium', but not when S is certain that the object is either 'tall' or 'medium'. In a YN situation, (in which S is asked 'Is this object 'tall OR medium'), the label 'tall OR medium' applies to all objects whose height lies in the union of the height intervals which apply to 'tall' and to 'medium' respectively. Consequently, there exist two radically different membership curves for 'tall OR medium'. The curve which refers to the voluntary LB situation is much narrower than that which refers to the YN situation. This effect is much bigger than the additional difference between the two membership curves for the conjunctive label in YN situations referring to SIM and RR composite experiments respectively (see Hisdal 1986b, fuzziness #10). Labels with connectives will be discussed in detail in Hisdal (1988b). The formulas for composite labels with YN reference are summarized in Hisdal (1988c).

The second goal of this paper is to show that the interpretation of a label λ, and the membership curve of the label, depend upon the label set Λ to which the subject refers. This set must be identified, or guessed at by the listener (or by the experimenter E in a more formal experiment) in order to interpret the meaning which the subject S attaches to λ or its membership value.

E.g., consider the situation in which S refers the label $\lambda =$ tall to the label set

$$\Lambda 1 = \{\text{small, medium, tall}\} , \tag{1}$$

versus the situation in which he refers to the label set

$$\Lambda 2 = \{\text{small, tall}\} . \tag{2}$$

An object whose height lies in the upper medium range of height values will be considered by S to be 'medium' when he refers to $\Lambda 1$, and to be 'tall' when he refers to $\Lambda 2$. This statement applies to both LB and YN situations. The labeling in these situations will influence the shape of the membership curves elicited in a MU experiment; with the result that the S-shaped membership curve of 'tall' which refers to $\Lambda 1$ is displaced towards larger height values as compared with the one which refers to $\Lambda 2$. Since the reference label set serves the purpose of a standard in the interpretation of a semantic experiment, it is important to know what combinations of labels are acceptable as label sets to which a semantic experiment refers.

The four types of semantic experiment are discussed in section 2, and the subject of legal label sets in section 3.

A third goal of the paper is to present the first two, or the 'Threshold', 'Error' assumptions of the TEE model ('TEE' stands for 'Threshold, Error, assumption of Equivalence'). These assumptions are presented in section 4, and the implications from the first three assumptions concerning the fuzzy set 'one-minus' postulate for the negation (and more generally the summing up to 1 postulate for the grades of membership of one point of the attribute universe in the different elements of a label set) in section 5. In appendix A1, the negation and antonyms are discussed again in more detail, and the TEE model predictions are compared with the experimental results of Hersh & Caramazza, and Norwich & Turksen. Appendix A1 shows also that formal experiments concerning the negation may be very difficult to carry out in a consistent way due to the existence of several higher level interpretations of the negation in natural language.

Section 6 presents the LB-YN assumption and the simple TEE model assumption for the VERY modifier which results in a displacement of the membership curve along the abcissa axis, not in the 'mu-square' law. An overview concerning the situation- and context-dependence of adjective labels is given in section 7.

Some of the most important terminology is laid down in section 2, definition 1. The remaining terminology is summarized in fig. 1.

To limit the scope of the paper we refer mostly to fuzziness # 1, i.e. fuzziness due to variable conditions of observation. Fuzziness # 2 and 3 have already been treated summarily in Hisdal (1986b).

2. The Four Semantic Experiments

In this section we define four semantic experiments. These are LB, YN, LB-MU and YN-MU experiments. The symbols stand for 'labeling', 'yes-no', 'grade of membership referring to LB-', and 'grade of membership referring to YN-' situations repsectively. The main emphasis in our definitions is to provide a semantic environment for the subject which enables him to identify unambiguously the situation in everyday life to which the experiment and the label λ refer with respect to: 1) A clear differentiation between an LB situation in which the subject chooses the label λ voluntarily, versus a YN situation in which the label is prespecified by the interrogator. 2) A clear differentiation between MU experiments refering to LB situations versus MU experiments referring to YN situations. 3) An identification of the reference label set to which every semantic experiment must refer if it is to give consistent results.

We do not take up the question of scaling, and for MU experiments we assume that the subject is instructed to specify a membership value in the interval [0, 1]; or in $\{0, \ldots, 1\}$ in the case of the prespecification of quantized membership values, e.g., $\{0,\ 0.1,\ 0.2,\ \ldots,\ 1\}$. Such a mapping from objects or attribute values to [0, 1], has always been assumed by Zadeh (see, e.g., Zadeh,1973). However, it seems that human beings with a minimum of mathematical knowledge have little difficulty in mapping their results on other prespecified intervals; such as on the much-used percentage interval [0, 100]; or on a geometrically represented interval in the form of a line-segment or of an angular segment. No matter what type of interval was used in the experiments, the experimentalists have always converted their results for MU experiments to the interval [0, 1] (Hersh and Caramazza (1976), Norwich and Turksen (1982a), Wallsten, Budescu, Rapoport, Zwick & Forsyth (1985), Zimmermann and Zysno (1980)).

The upper bounds of the different types of prespecified intervals for the range of the membership function must always be assumed to correspond to each other, and similarly for the lower bounds. When no object, or no exact atribute value u^{ex}, is assigned a membership value equal to the upper bound, then we have a subnormal fuzzy set according to Zadeh's terminology. Norwich and Turksen (1982a,b) normalize the membership functions in the case of subnormality in order to ensure that there always exist objects or exact attribute values with membership values 0 and 1. We have already discussed the meaning of subnormality in the TEE model, and why no normalization should be performed on subnormal fuzzy sets in this model (Hisdal (1986a), difficulty 16b; also (1986b), fuzziness #2a). The meaning of subnormal fuzzy sets is discussed again in Hisdal (1988a), sect. 3, consequence 7.

Another question which we do not take up is how the experimenter can provide a physical environment for the subject which simulates that of everyday life. E.g., in everyday life the height of a person is, in most cases, judged against some standard of comparison; such as the height of other persons or of a standard door opening. In an experimental situation, such standards should therefore also be provided in the surroundings of the object in connection with a label such as 'tall' or 'large' because the eye can judge dimensions only on a relative basis. In contrast, for a label such as 'old', the presence of a standard of comparison is of minor importance. Wrinkles, or color of hair, or baldness can be judged quite well also in the absence of such standards (see also Hisdal 1986a, end section 1, in this connection).

The following preparatory definition describes the elements and definitions which are common to all three types of semantic experiment.

Definition 1 of 11 items and subdefinitions which are common to LB, YN and MU experiments.

1) An experimenter E who plans the experiment, gives instructions to the subject S, performs the E-experiment of item 6, and analyses the semantic experiment.

2) A set of objects $OB = \{ob\}$ chosen at random by E from the context dependent class of objects to which the label λ refers. (E.g., from the class of adult, male human-beings when the subject is instructed to answer the question 'Is this man tall?' for every object ob.) When the dependence of the answers on the set of conditions of observation is investigated, then E must ensure that the elements of OB are randomized with respect to conditions of observation (see Hisdal (1986b, sect.1)).

3) A single subject S who performs E's instructions on each object $ob \in OB$ in turn.

4) A set of instructions given by the experimenter E to the subject S,

depending on the type of semantic experiment. These instructions are described in definitions 2–5 below. For each of the four types of semantic experiment they result in an 'answer-value' given by S for each object *ob*. (A Y or N value concerning the object being λ for a YN experiment; a label $\lambda \in \Lambda$ for an LB experiment; and a membership-value μ_λ for both types of MU experiment.) It is the task of the experimenter to make a list of the objects, of their u^{ex} values, and of the answer values which S assigns to them (see items 6, 7).

Note that the answer value for a given object *ob* refers to one specific experiment. In another LB or YN experiment, in which objects are rerandomized with respect to conditions of observation, the subject's estimate u of the attribute value of the same object will, in general, differ from the estimate in the first experiment. It may then happen that S assigns a different answer value to *ob*. As we shall see in Hisdal (1988a), it is the recognition of this fact by S which gives rise to the grade of membership concept according to the TEE model.

In the instruction which E gives to S we use, in defs. 2–5, the notation '*ob*' for the pertinent object. Whenever necessary, '*ob*' should not be replaced by the word 'object', but by a nounphrase which specifies the context in which S is to give his answers. E.g., in connection with linguistic height labels, '*ob*' might be replaced by 'man' or 'woman' or 'six year old girl'.

5) The attribute A to which the experiment refers (e.g., A =height for λ =tall), and the universe of attribute values. The universe of numerical attribute values is denoted by U^{ex} (e.g., U^{ex} =[0,250) cm) when it refers to the values measured in the exact E-experiment of the next item. When it refers to the values estimated by S (see def. 10), it is denoted by U . The elements of these universes are denoted by u^{ex} and u respectively, often with the subscript ' i ', $i = 1, \dots, I$. In the following we shall assume that the two universes consist of the same set of numerical values. In our examples, we use a 'quantized', continuous universe; in the sense that, e.g., u^{ex} =165 cm should be interpreted as $u^{ex} \in [160, 170)$ cm. The subset $\Delta u^{ex} = [160, 170)$ cm is called the 'numerical quantization interval' for u^{ex} =165 cm. δu^{ex} =(170–160) cm=10 cm is called the size of the numerical quantization interval. In our examples it is a constant, independent of u^{ex} .

The attribute A can also be 'multidimensional'. This means that it consists of several subattributes with a corresponding multidimensional universe

$$A = A^1, A^2, \dots \;, \qquad U = U^1 \times U^2 \times \dots \;, \tag{3}$$

and similarly for U^{ex}. E.g., for λ =slim we can have A^1 =height, A^2 =weight, $U^1 = \{5 \text{ cm}, \dots, 245 \text{ cm}\}$, $U^2 = \{5 \text{ kg}, \dots, 175 \text{ kg}\}$. Concerning unclarity with respect to the subattributes pertaining to a given linguistic label, see Hisdal (1986b), fuzziness #2a, 2b.

6) The 'exact experimenter experiment' or the 'E-experiment' for short. This experiment is a part of the semantic experiment. However, it is carried out solely by the experimenter E, and is thus independent of the subject S. E measures and makes a note of the exact attribute-value u^{ex} of each $ob \in OB$. This value is, in general, not communicated to the subject.

7) The I subsets $OB_i \subset OB$, $i = 1, \dots, I$. Let

$$U^{ex} = \{u_i^{ex}\}, \qquad i = 1, \dots, I \;, \tag{4}$$

be the universe of attribute values used by E in the E-experiment. When the subject S has finished giving his answers, then E uses the results of the E-experiment of item 6 to partition the object set OB into I disjoint subsets OB_i,

$$OB = \cup_{i=1}^{I} OB_i, \qquad OB_i \cap OB_{i'} = \emptyset \quad \text{for} \;\; i \neq i', \qquad i, i' = 1, \dots, I \;. \tag{5}$$

Each subset OB_i consists of those objects whose exact attribute value is $u_i^{ex}(\pm \delta u_i^{ex}/2$, see item (5)). The cardinality of (number of elements in) OB_i will be denoted by $card_i$, and the cardinality of OB by $card$,

$$\sum_{i=1}^{I} card_i = card \;. \tag{6}$$

The purpose of the partition (5) is to estimate the probability P_i that the subject will assign a particular answer value to objects of exact attribute value u_i^{ex}; e.g., the probability of 'Y' answers in a YN experiment for objects of this attribute value. Or the probability that S will assign a particular μ_λ-value to these objects in a MU experiment. Note that the latter probability has no connection with a probabilistic model for grades of membership. The elicitation of membership values in a nonexact MU experiment will always result in a spread of the μ_λ-values for objects of a given exact attribute value. This effect has been named 'stochastic fuzziness' by Norwich & Turksen (1982a) who analyze it in detail. We have analyzed it under fuzziness #1b in Hisdal (1986b, sects. 1, 3 and appendix).

Let n_i be the number of elements in OB_i which have been assigned a particular answer value by S in a semantic experiment. The estimate by E of the probability of this answer value, as computed from the specific experiment, is then given by

$$P_i = n_i / card_i \ . \tag{7}$$

To simplify the notation we will, in the following, denote the probabilities, and their estimates by E from the specific experiment, by the same letter P. Equality between the true probabilities and their estimates holds within the limits of statistical fluctuations. The bigger the cardinality of OB_i, the better is the estimate of the probability.

8) $P(u^{ex})$, the unqualified probability distribution over U^{ex} (e.g. the distribution over height of the population of objects). $P(u_i^{ex})$ is the probability that a randomly chosen element of the object set *OB* of item 2 will have the exact attribute value $u_i^{ex}(\pm\delta u_i^{ex}/2)$. It is 'unqualified' by a label such as 'tall'. The unqualified distribution can either be assumed to be known in advance, or it can be found from the E-experiment of item 6, using the formula,

$$P(u_i^{ex}) = card_i / card \ , \qquad i = 1, \ldots, I \ . \tag{8}$$

The remark following eqn (7) holds also for the distribution found from (8).

$P(u^{ex})$ is independent of the subject S; in contrast to all the other distributions defined in the TEE model which are subjective for S. They can, of course, be averaged over many subjects.

9) A label set $\Lambda = \{\lambda_l\}$, $l = 1, \ldots, L$, to which S refers his answers (see sect. 1 concerning the necessity of this item). Λ can be prespecified by E to S. It must then be ascertained that S accepts it as being complete and nonredundant *in the context of an LB experiment.* Completeness of Λ means that when S performs an LB experiment, then he can find a label $\lambda_l \in \Lambda$ for every object in the class of context dependent objects. Nonredundancy means that there are no objects for which the subject insists on the assignment of more than one element of Λ. (We shall, however, see in Hisdal (1988b) that more sophisticated label sets may contain elements such as 'tall OR medium'.)

Λ is called a legal reference label set for S in connection with any of the four semantic experiments iff S accepts it as being complete and nonredundant in connection with an LB experiment. A semantic experiment which refers to a label set that S accepts as a legal one is called a legal semantic experiment. Unless something else is mentioned, we will always assume in the TEE model papers that the semantic experiment under discussion is a legal one.

If there exists at least one subject for whom Λ is a legal label set, then Λ is called a potentially legal label set.

Λ is called YN-nonredundant iff there does not exist any object in the context dependent class of objects for which the subject gives a Y answer both concerning the question 'Is *ob* λ_1?' and concerning the question 'Is *ob* λ_2?', where $\lambda_1, \lambda_2 \in \Lambda$ and $\lambda_1 \neq \lambda_2$. (The two questions concerning the single object are assumed to be asked SIMultaneously so that the subject answers them on the basis of the same estimated attribute value u.) As an illustration, the label set of equation (16) below, which contains both the element 'tall' and the element 'VERY tall', is nonredundant but not YN-nonredundant.

Alternatively, instead of prespecifying the reference label set Λ to S, the experimenter E can elicit Λ by noting the labels used by S in a great number of cases. For LB experiments this can be done as part of the experiment itself. For YN and MU experiments, the elicitation of Λ must be carried out in a separate, preparatory LB experiment.

In defs. 2–5 below we assume that Λ is prespecified to S.

10) An exact semantic experiment. The semantic experiment is called exact when S measures or is told the exact attribute value u^{ex} of each object $ob \in OB$. Otherwise the experiment is called nonexact. In an exact experiment we have that $u = u^{ex}$, where u is the subject's estimate of the attribute value of the object (see def. 10).

11) The set of conditions of observation of the experiment. (See Hisdal (1986b, sect. 1). Also Hisdal (1988c, Turksen's criticism in section 2.8).) The elements of OB, item 2, are assumed to be assigned at random to the points of this set according to a given probability distribution; resulting in a final error function $P(u|u^{ex})$. This is the probability that the subject will estimate the attribute value of an object to be u, when the true attribute value (measured by the *experimenter*

in an *exact* experiment) is u^{ex}. If the set of conditions of observation consists of a single point, then we have 'constant conditions of observation'. If this single point corresponds to an exact observation or measurement, then we have exact conditions of observation, and an exact experiment, item 10. The error function reduces to a delta function centered on $u = u^{ex}$ in this case.

end def. 1

We now go over to defining the specific instructions which E gives to S for each of the different types of semantic experiment.

Definition 2. An LB (labeling) experiment. The experimenter E chooses a label set Λ,

$$\Lambda = \{\lambda_1, \ldots, \lambda_l, \ldots, \lambda_L\} , \tag{9}$$

and a set of objects *OB*. He presents S with each element *ob* of *OB* in turn and gives him the following instruction: "Imagine that you are talking to a person X who cannot observe *ob*. You wish to describe *ob* to him by saying

$$ob \text{ is } \lambda_l \qquad \text{(e.g., This man is small.)} \tag{10}$$

where λ_l belongs to the set (9) (e.g., the set of eqn (1)). Choose the element of Λ which you consider to be appropriate to *ob* when substituted for λ_l in (10)."

Alternatively, (10) may be considered to be the answer to the question

$$\text{How } \lambda_p \text{ is } ob\,? \qquad \text{(How tall is this man?)} \tag{11}$$

asked by X. λ_p is the primary label pertaining to A (see def. 6).

end def. 2

Typical mathematical terms which we use in the instructions to the subject in defs. 2–5 should be converted to terms used in everyday discourse. E.g., the word 'set' should be replaced by 'list'.

Definition 3. A YN (yes-no) experiment. The instruction to S for a YN experiment is: "Please answer the question

$$\text{Is } ob\ \lambda_{spec}\,? \qquad \text{(Is this man VERY young?)} , \tag{12}$$

where the 'specified label' λ_{spec} refers to the set Λ of eqn (9). Your answer should be either 'Y' or 'N'." When the answer is 'Y', then we say that 'the label λ_{spec} has been assigned to *ob* in the YN experiment'; or that *ob* has been assigned the label 'Y- λ_{spec}'; and similarly 'N- λ_{spec}' for a 'N' answer.

Any element of Λ is a 'legal specified label', in the sense that it may be substituted for λ_{spec} in (12). In addition, we shall define some other legal specified labels in due course. The most important of these are two or more elements of Λ connected by OR or AND connectives. Negated specified labels are discussed in appendix A1 which concludes that the use of such labels is not to be recommended for formal semantic experiments.

end def. 3

Definition 4. An LB-MU experiment (grade of membership experiment referring to an LB experiment). E gives the following instruction to S: "Imagine that you are talking to a person X who does not see the object *ob*. Imagine also that somebody has described *ob* by the sentence (10), where λ_l is one of the elements of the set Λ of eqn (9). For each of the L labels which belong to Λ, tell X the degree μ_{λ_l}, $\mu_{\lambda_l} \in [0,1]$, to which you consider that the description (10) is the appropriate one for *ob*."

Definition 5. A YN-MU experiment (MU experiment referring to a YN experiment). Let λ_{spec} be a legal specified label (see def. 3). There exist two YN-MU experiments for λ_{spec}, one concerning *Y-* λ_{spec}, and one concerning *N-* λ_{spec}. For the *Y-* λ_{spec} case E gives the following instruction to S: " Imagine that a person X who does not see *ob* asks you the question (12), where λ_{spec} refers to the set Λ of eqn (9). Tell X the numerical degree to which you consider that a 'Y' answer to this question is correct. The degree should be 1 when you are certain that a 'Y' answer is correct, and 0 when you are certain that a 'N' answer is correct. In all other cases it should be a number between 0 and 1."

For a YN-MU experiment concerning *N-* λ_{spec}, the middle part of the above instruction must be changed to: "Tell X the numerical degree to which you consider that a 'N' answer to this question is correct. The degree should be 1 when you are certain that a 'N' answer is correct, and 0 when you are certain that a 'Y' answer is correct."

In the following we shall refer to these degrees as $\mu_{Y-\lambda_{spec}}$ and $\mu_{N-\lambda_{spec}}$ respectively. Sometimes we will refer to the former simply as $\mu_{\lambda_{spec}}$.

3. Legal Label Sets

In the previous two sections we emphasized the importance of providing an experimental environment for the subject which will enable him 1) To identify the experimental situation as to whether it refers to an LB or YN situation of everyday life. 2) To identify the label set Λ to which his answers are to refer. If the semantic experiment is unclear with respect to one or both of these items, then we cannot expect a consistent behaviour on the part of the subject. Neither can we expect qualitative interconsistency between the answers of different subjects.

LB situations, in which a subject describes an object by choosing a label from his stock of labels, are probably more common in everyday discourse than YN situations. In spite of that, they have largely been neglected on the experimental side. This is probably due to the difficulty of identifying the label set to which a subject refers; or to the lack of attempts to define label sets which, at least some subjects, can accept as being complete and nonredundant (see item 9 of def. 1).

We therefore define in this section some simple label sets which we believe, on the basis of self-experimentation, to be potentially legal. We make no attempt to exhaust all the different possibilities for potentially legal label sets offered by the English language. Our starting point is the 'twin label set' for attributes whose linguistic values are adjectives.

Definition 6. The twin label set. Very many attributes have two linguistic values, each of which consists of a single word which is an adjective. The two words, which are antonyms of each other, are either completely different, or one of them is a concatenation of 'un' (or some similar prefix) with the other. We shall call these two labels 'twin labels', and a label set consisting solely of these two labels a 'twin label set'. Examples of twin label sets are: {small, tall}, {young, old}, {slow, fast}, {cheap, expensive}, {light, dark}, {slim, obese}, {bad, good}, {ugly, beautiful}, {pleasant, unpleasant}, {probable, improbable}.

For most one-dimensional attributes, one of the twin labels has a special function. It is used in questions of the type of (11) (we do not say 'How small is John?', but 'How tall is John?'). This label ('tall') will be called the primary one for the given attribute ('height'), and will be denoted by λ_p . The other label will be called the secondary twin label and denoted by λ_s . The general notation for a twin label set pertaining to an attribute A is thus,

$$\Lambda = \{\lambda_s, \lambda_p\} \ . \tag{13}$$

We note two exceptions to the linguistic exclusiveness of the twin labels. One is the temperature attribute for which we have, in English, three different non-compound adjectives to characterize three intervals. Namely 'cold', 'warm', 'hot'. In this case we can stretch our definition, and consider {cold, warm} to be the basic twin set. 'hot' can then be considered to be semantically equivalent to 'VERY warm', and 'VERY hot' to 'VERY VERY warm' (see sect. 6).

A more fundamental exception to the twin set situation occurs for the hue attribute, for which no twin set exists. This is due to the fact that the hue sensation of the eye is not based on the physical wave length attribute of the light, but on a combination of the responses of three types of receptor, each of which responds to a wide band in the red, green and blue regions of the spectrum respectively (Boynton, 1984).

The following definition of a triple label set has no connection with the just-mentioned exceptions to the linguistic exclusiveness of the twin labels.

Definition 7. The triple label set. Experimental evidence on antonyms (see appendix A1) shows indirectly that most subjects prefer the use of the triple label set to the twin set. The triple set contains the two twin labels. In addition it contains a label λ_m which applies to the medium region of attribute values. Its linguistic name for different attributes has usually some common element such as 'medium'; or 'middle' in 'middle-aged'. Eqn (1) is an example of a triple label set. The general notation for such a set is,

$$\Lambda = \{\lambda_s, \ \lambda_m, \ \lambda_p\} \ . \tag{14}$$

Definition 8. The quintuple label set. This set contains the modifications of λ_s and λ_p by 'VERY' in addition to the elements of the triple set,

$$\Lambda = \{VERY \ \lambda_s, \ \lambda_s, \ \lambda_m, \ \lambda_p, \ VERY \ \lambda_p\} \ . \tag{15}$$

An example of such a set is,

$$\Lambda = \{\text{VERY small, small, medium, tall, VERY tall}\} \ . \tag{16}$$

Definition 9 of the assumption of legal and illegal label sets and YN nonredundancy. This assumption says that the twin, triple and quintuple label sets are all potentially legal label sets (see item 9 of def. 1). The number of subjects for whom the three sets are legal ones probably increases from twin sets to triple sets and further to quintuple sets. If one of the twin labels is removed from any of these three sets, then the resulting label set is no longer complete, and is therefore an illegal one for every subject. Here we assume, of course, that the subject does not assign new meanings to the linguistic labels which deviate from the meanings in everyday discourse. Such a reinterpretation of everyday words is not at all uncommon in a given scientific context. However, since we investigate here the working and use of logic in natural language, we cannot accept such deviant interpretations.

Furthermore we shall assume that for those subjects, for whom the label sets (13), (14) are legal ones, and therefore nonredundant, they are also YN-nonredundant. In contrast, the label set (15) is assumed to be YN-redundant (see item 9 of def. 1 and def. 15).

end def. 9

Our guess is that extremal labels such as 'small' and 'tall' are used by subjects to signify that the object's attribute value lies outside the attribute-interval of the majority of objects; and that therefore the medium label of the triple set is the one which most subjects will attach to the majority of objects. Unfortunately, the medium-type of labels have, up to now, largely been neglected by the experimentalists.

4. The First Two Assumptions of the TEE Model and Interpretational versus Operational Definitions

In order to explain the results of semantic experiments, the TEE model makes several assumptions concerning the meaning of the labels and of the membership values assigned to objects. The three main assumptions are numbered 1, 2 and 3 respectively. The first two of these are presented in the present section. The important third assumption, or LB,YN-MU assumption of equivalence, follows in the next paper (Hisdal, 1988a).

Assuming that the first two assumptions are true, we then discuss two types of definitions in the TEE model, interpretational and operational ones. The former refer to quantization intervals in the universe U of estimated attribute values. The operational definitions refer to sets of labeled objects. Both types of definitions are significant; the interpretational ones because they are the last link in the chain which determines the subject's answer value in any semantic experiment.

The operational definitions are the more sophisticated ones. A subject who uses them correctly has the ability to estimate the fuzzifying effect of different types of uncertainty; in the sense that objects of the same exact attribute value u^{ex} do not necessarily give rise to the same answer in a semantic experiment. The grade of membership concept itself is thus based on a combination of an interpretational and an operational definition. The subject estimates the fraction of objects of a given u^{ex} which would be assigned the label λ in an LB or YN situation, in the presence of different sources of fuzziness (Hisdal, 1986b, fuzziness #1a, 2a, 3a; Hisdal (1988a)). Other definitions can be purely operational or purely interpretational. Complete and nonredundant label sets, which were defined operationally in def. 1, item 9, are redefined interpretationally in def. 14.

If we believe at all in the possibility of a theory of artificial intelligence for natural language, and of computer systems which simulate the processing of language performed by humans, then we must also believe that the meaning of a term is determined by the result of a procedure carried out by the person who uses the term. In the final analysis this procedure must involve information received by our senses, and processed in our brains. Even a so-called abstract concept like 'a good person' is connected with the observation of the person's behaviour in relation to other persons. A concept like 'mathematical theory' is meaningless unless the symbols of the theory can be distinguished by our senses (sight, hearing, or possibly touch in the case of a blind person) and assigned a specific processing procedure in our brains.

We shall here start with terms (such as 'tall') describing values of one-dimensional attributes, and then say a few words about the multidimensional case. The following is a superficial summary of the first two assumptions of the TEE model which are presented in more detail below.

Summary of the first two assumptions of the TEE model:
The subject's answer in any semantic experiment (LB or YN or MU) is a function of his estimate u of the attribute value of the object. Every subject constructs intervals in the universe U of estimated attribute values which correspond to each of the possible answers in an LB or YN experiment.

For example, in the figures to the TEE model papers, we have assumed that the 'quantization intervals' of a given subject are,

$$\Delta u_{small} = \{105, 115, \ldots, 135\} \text{ cm} , \quad \Delta u_{medium} = \{145, 155, 165\} \text{ cm} ,$$
$$\Delta u_{tall} = \{175, 185, \ldots\} \text{ cm} , \tag{17}$$

for an LB experiment concerning the height of adult women, and referring to the label set (1). For a YN experiment with λ_{spec}=tall, the last quantization interval in (17) is the one corresponding to 'Y' answers. For MU experiments, the subject's answer is also a function of his estimate u of the height value of the object. Namely the $\mu_\lambda(u)$ membership function which he has stored in his brain. The original meaning and construction of this function are discussed in the next paper (Hisdal, 1988a).

A more precise formulation of the first two assumptions of the TEE model, as well as some details concerning their justification, follow below.

In the following we assume that the estimated and exact attribute universes U and U^{ex} are ordered sets. To simplify the notation, we will assign numerical values to the points of these sets, although these values will usually only be relative ones in the subject's mind; such as a comparison of a person's height with the height of other persons; or with the height of a door opening.

Let us start by assuming that the subject performs an exact experiment in which he measures u^{ex}, the exact attribute value of each object or person with which he is presented (item 10, def. 1), and assigns a label, or a YN answer, or a MU answer to each object. The TEE model then assumes that the subject's answer will be a function of the measured attribute value u^{ex} for all three types of experiment. This is quite in keeping with present-day fuzzy set theory which assumes that the grade of membership of a given label is a function of u^{ex}.

Just for the moment we will skip over the question of what procedure the subject uses to give his answer, once he knows the value of u^{ex}. We will only suppose that some such procedure must exist.

When the subject does not have the opportunity to measure the object's u^{ex}-value, and this is the usual situation in everyday discourse, then the TEE model assumes that he will make an estimate of the object's attribute value. To give his answer, he will use the same procedure as in the case of the exact experiment, the only difference being that he replaces u^{ex} by his estimate u of the object's attribute value. In summary, we have the following:

Definition 10 of the first assumption of the TEE model, or the assumption of the intermediate information processing step: A subject who performs an LB or YN or MU experiment estimates the object's attribute value, and bases his answer on the estimated attribute value u. u is equal to u^{ex} in an exact experiment (item 10, def. 1).

The second assumption of the TEE model concerns the procedure which a subject uses to give his answers in the case of an exact LB or YN experiment. As far as I can see, there exists only one procedure which the subject can use:

Definition 11 of the second assumption of the TEE model or the assumption of nonfuzzy thresholds in connection with exact LB and YN experiments (item 10, def. 1 and defs. 2, 3): In an exact LB experiment referring to the label set Λ, the subject partitions the universe U^{ex} of exact attribute values into L 'quantization intervals' Δu_{λ_l}, $l = 1, \ldots, L$, each of which is labeled by one of the linguistic labels λ_l of eqn (9). For our quantized, one-dimensional U^{ex} universe with numerical values u_i, $i = 1, \ldots, I$, we can then write

$$\Delta u_{\lambda_l} = [u_{\lambda_l,l},\ u_{\lambda_l,u}] \tag{18}$$

where $u_{\lambda_l,l}$, $u_{\lambda_l,u}$, the lower and upper bounds of the quantization interval, are nonfuzzy lower and upper thresholds respectively for λ_l . The subject assigns that label $\lambda_l \in \Lambda$ to the object for which

$$u^{ex} \in \Delta u_{\lambda_l} \ . \tag{19}$$

In the case of a YN experiment concerning λ_{spec} (see also def. 15), the subject partitions U^{ex} into two intervals for 'Y' and 'N' answers respectively,

$$\Delta u_{Y-\lambda_{spec}} \quad \text{and} \quad \Delta u_{N-\lambda_{spec}} \ . \tag{20}$$

We add two remarks concerning notation.

1) In the present definition, we have used the letter u for the quantized numerical values of U^{ex} . Indeed U^{ex} and U assume values in the same numerical universe.

2) For our quantized universes we choose the numerical values of the thresholds to lie midways between the biggest point of one quantization interval and the smallest point of the next higher quantization interval. In this way, the upper bound of the lower interval is numerically equal to the lower bound of the next higher interval in the notation of the right hand side of eqn (18); and both of them are equal to the nonfuzzy threshold value which, however, is not a quantization point. The right hand side of eqn (18) denotes the finite set of quantized points whose numerical values lie between the lower and upper bounds or thresholds for λ_l , e.g., the sets of eqn (17). For these the threshold between 'medium' and 'tall' is $u_{medium,u}, = u_{tall,l} = 170\,\text{cm}$. This is not a point of the quantized universe.

end def. 11

A combination of the second assumption with the first one gives us the following result:

Second Assumption of the TEE Model, combined with the First Assumption: When a subject performs an LB or YN experiment under exact or nonexact conditions of observation, his first step is to make an estimate u of the object's attribute value. In the LB experiment he assigns that label λ_l to the object for which

$$u \in \Delta u_{\lambda_l} \ , \tag{21}$$

where Δu_{λ_l} , the quantization interval for λ_l , is given by (18). In the YN experiment he assigns a 'Y' answer when

$$u \in \Delta u_{Y-\lambda_{spec}} \ , \tag{22}$$

and a 'N' answer otherwise. For exact conditions of observation (see item 10, def. 1), $u = u^{ex}$.

Definition 12 of t_{λ_l}, the threshold curve of λ_l or the likelihood distribution of λ_l over u. For an LB experiment, a function t_{λ_l} of the estimated attribute value u, defined by

$$t_{\lambda_l}(u) = P(\lambda_l \mid u) = \begin{cases} 1, & \text{if } u \in \Delta u_{\lambda_l}; \\ 0, & \text{otherwise,} \end{cases} \tag{23}$$

is called the threshold function of λ_l, or the likelihood distribution of λ_l over u. It can be interpreted as the probability that the subject will assign the label $\lambda_l \in \Lambda$ to an object whose attribute value he estimates to be equal to u. An analogous definition holds for the likelihood function of λ_{spec} in a YN experiment with λ_l in (23) replaced by λ_{spec}. We will use the name 'likelihood or threshold function of λ for both the LB and the YN functions. If we want to emphasize that the function refers to a YN experiment, we will also call it the threshold or likelihood function of Y- λ.

end def. 12

Fig. 2 shows the threshold function for 'tall' (full curve) and 'medium' (broken curve) respectively, using the quantization intervals of (17). In (Hisdal, 1988a) we show how these two curves are rounded off in a nonexact experiment when the abcissa axis represents u^{ex} instead of u. Furthermore we show that there is a close connection between these rounded-off 'step' and 'square pulse' shaped curves (for 'tall' and 'medium' respectively), and the corresponding S- and bell-shaped grade of membership curves elicited in a MU experiment.

Everything that we have said up to now in this section is unchanged in the case of a multidimensional attribute A (item 5, def. 1). The subset Δu_λ of U corresponding to a given label λ is assumed to be an interval in the multidimensional universe U, and is called the 'quantization interval for λ'; where an 'interval' is defined to be a connected subset of U in the case when U is a connected space (i.e. a 'continuous universe' in more plebeian terminology). An analogous assumption concerning quantization intervals holds for a quantized universe U. A precise definition of an 'interval' in a quantized multidimensional universe would take so much space, and require so much expertise in the terminology of topology, that we leave it out.

As an example, consider the attribute 'obeseness' in the two-dimensional universe of height $\times$ weight. The thresholds or partitioning boundaries between two linguistic values of obeseness (of a nonredundant label set) are approximately third degree curves centered at the origo (because weight is approximatley proportional to volume). The quantization interval Δu_λ for a given label is then the area between two such curves. A possible quantization interval for $\lambda =$ slim is shown by the shaded area in fig. 3. The unsatisfactory nature of the noninteractive AND of fuzzy set theory (e.g., for the representation of 'slim' as 'tall AND light') and the interpretation of a multidimensional concept in the TEE model, are discussed in somewhat more detail in Hisdal (1984b, pp. 5-8 and 12).

Definition 13 of operational and interpretational definitions. We shall call a definition which makes use solely of the labeling results of objects in a given semantic experiment an 'operational definition'. An operational definition which refers to partitioning, equality or subsets of OB has a corresponding 'interpretational definition' which refers to partitioning, equality or subsets (quantization intervals) of the universe U of estimated atribute values. The latter type of definitions will be called 'interpretational definitions'. For example, the definitions of a complete and of a nonredundant label set (def. 1, item 9) are operational definitions. Definition 14 below is the corresponding interpretational one. In this case the two definitions turn out to be equivalent. In contrast, we shall see in part 1.5 of this series that the operational and interpretational definitions of subset and disjointness relations for labels are equivalent only in the SIM case.

Definition 14. Interpretational definition of a complete, a nonredundant and a legal label set. A label set $\Lambda = \{\lambda_1, \ldots, \lambda_l, \ldots, \lambda_L\}$ is complete iff the union of the quantization intervals of the L labels of Λ (as pertaining to an LB experiment) is equal to the universe of estimated attribute values,

$$\cup_{l=1}^{L} \Delta u_{\lambda_l} = U \ . \tag{24}$$

Λ is nonredundant iff the quantization intervals are disjoint,

$$\Delta u_{\lambda_l} \cap \Delta u_{\lambda_{l'}} = \emptyset \qquad \text{for} \quad l \neq l' \quad \text{and} \quad l, l' = 1, \ldots, L \ . \tag{25}$$

Assume that the subject performs L YN-experiments, all of which refer to the label set Λ. The specified label of the l-th experiment is $\lambda_{spec} = \lambda_l, \quad l = 1, \ldots, L$. Λ is called YN-nonredundant iff the quantization intervals for Y-λ_l, $l = 1, \ldots, L$ are all mutually disjoint.

end def. 14

The disjointness of the quantization intervals in U, required by eqn (25), may seem at first sight to contradict the philosophy of fuzzy sets. Note, however, that disjointness in the universe U implies, in general, an overlap of neighboring intervals in U^{ex}. This subject is discussed again in part 1.5 of this series.

According to theorem 1 below, the above interpretational definitions are equivalent to the operational definitions of item 9 in definition 1. Λ is therefore a legal reference label set for S in connection with any of the four semantic experiments also iff it is interpretationally complete and nonredundant.

Theorem 1. The interpretational definitions 14 of a complete, a nonredundant and a YN-nonredundant label set are equivalent to the operational definitions, item 9 of def. 1.

Proof of theorem 1: Let OB_{λ_l}, $l = 1, \ldots, L$, be the subset of OB to which S assigns the label λ_l in an LB experiment. From def. 1, item 9, it then follows that the following two requirements are necessary and sufficient conditions for operational completeness and nonredundancy of Λ respectively:

$$\cup_{l=1}^{L} OB_{\lambda_l} = OB \ , \quad \text{(operational completeness)} \tag{26}$$

$$OB_{\lambda_l} \cap OB_{\lambda'_l} = \emptyset \quad \text{for} \quad l \neq l' \quad \text{and} \quad l, l' = 1, \ldots, L \ . \quad \text{(operational nonredundancy)} \tag{27}$$

Equations (26),(27) must hold for every sample OB from the context dependent class of objects.

But according to the first and second assumptions of the TEE model, those and only those objects are assigned the label λ_l for which the subject's estimate u of the object's attribute value satisfies $u \in \Delta u_{\lambda_l}$. It then follows that equations (24),(25) are equivalent to equations (26),(27) respectively. The proof for a YN-nonredundant label set is similar.

5. The Summation Theorem and the 'One-Minus' Theorem for the Negation

There exist two important postulates of present-day fuzzy set theory which do not fit into the max-min framework with which this theory is often identified. One of these is Zadeh's postulate for the negation which says that the grade of membership of 'NOT λ' is equal to one minus the grade of membership of λ for a given value of u^{ex} (Zadeh, 1973 p. 32). The other postulate is due to Ruspini (1969). It says that the sum of the grades of membership in different labels (clusters) for a given value of u^{ex} (where U^{ex} is, in general, a multidimensional attribute universe) is equal to 1. This postulate has been retained by all workers in fuzzy clustering algorithms from 1969 onwards. In his 1969 paper Ruspini interprets grades of membership as probabilities, but he has now changed his opinion (Ruspini, 1985). I hope that the more detailed TEE model theory will convince him that his original interpretation is both meaningful and useful.

In the TEE model these two postulates are converted to theorems which follow from the first three assumptions of this model concerning the assignment by humans of linguistic labels and membership values to objects.

Here we shall formulate and derive the analogous theorem for likelihoods $P(\lambda|u^{ex})$. In the next paper (Hisdal, 1988a, defs. 2, 3) we present the third, or LB,YN-MU assumption of the TEE model according to which grades of membership are estimates by the subject of LB or YN likelihoods. Under the conditions specified in that paper, $P(\lambda|u^{ex})$ can therefore be replaced in all the formulas of this section by $\mu_\lambda(u^{ex})$. When the replacement of likelihoods by grades of membership is carried out in the equations of this section, then the proofs of theorems 2 and 3 below represent derivations in the TEE model of Ruspini's summation postulate and of Zadeh's 'one-minus' postulate for the negation respectively..

Theorem 2. The summing up to 1 theorem for likelihoods. Consider an LB experiment referring to the legal label set $\Lambda = \{\lambda_1, \dots, \lambda_l, \dots, \lambda_L\}$. The theorem says that the following two equations hold for LB experiments performed under nonexact or exact conditions of observation,

$$\sum_{l=1}^{L} P(\lambda_l|u^{ex}) = 1 \qquad \forall u^{ex} \; , \tag{28}$$

$$\sum_{l=1}^{L} P(\lambda_l|u) = 1 \qquad \forall u \ . \tag{29}$$

Note that irrespective of the value of the estimated attribute u, $P(\lambda_l|u)$ is always equal to 1 for one specific label $\lambda_l = \lambda_{l'}$, and to 0 for all the other labels. In contrast, we shall show in Hisdal (1987a) that $P(\lambda_l|u^{ex})$ can assume values between 0 and 1.

Equations (28),(29) follow from the consideration that for a given estimated or exact attribute value of the object, it is always assigned one of the L labels by S. For a more formal proof, we note that an LB experiment can be considered to be a statistical experiment in the three-dimensional universe $U \times U^{ex} \times \Lambda$. An outcome of an LB experiment thus consists of an object *ob* which is mapped upon a point (u, u^{ex}, λ_l) of this universe. u^{ex} is measured by the experimenter, u and λ_l are estimated and assigned by the subject S. These three variables are highly dependent. Equations (28),(29) then simply express the 'summing up to 1' property of conditional probabilities. Equation (28) refers to the marginal probability of λ_l in the universe $\Lambda \times U$ for a given value of u^{ex}. Equation (29) refers to the marginal probability of λ_l in the universe $\Lambda \times U^{ex}$ for a given value of u.

Theorem 3. The 'one-minus' theorem for the likelihood of λ and its negation. The theorem says that in a YN experiment, the sum of the probabilities of a Y and a N answer respectively is equal to 1 for a given attribute value, and consequently

$$P(N\text{-}\lambda_{spec} \mid u^{ex}) = 1 - P(Y\text{-}\lambda_{spec} \mid u^{ex}) \ , \tag{30}$$

$$P(N\text{-}\lambda_{spec} \mid u) = 1 - P(Y\text{-}\lambda_{spec} \mid u) \ . \tag{31}$$

The proof of theorem 3 is similar to that of theorem 2, except that we now work in the universe $U \times U^{ex} \times YN$, where $YN = \{Y, N\}$ is the universe consisting of the two answers of a YN experiment.

In appendix A1 we compare experimental results with the 'one-minus' theorem and show that natural language uses other interpretations of the negation all of which are, however, built on top of the above basic interpretation which follows from considerations of a YN experiment. The basic interpretation is defined more formally in appendix A1, def. A1.

6. The LB-YN Assumption and the VERY Modifier

We introduce two simple new assumptions in this section. The assumption of def. 15 concerns the connection between LB and YN experiments, and that of def. 17 concerns the VERY modifier. Both assumptions assert certain relations between the quantization intervals of different labels in the universe U of *estimated* attribute values, as used by a single subject S. It is then shown that the TEE model assumption of def. 17 predicts the effect of the VERY modifier (on extremal concepts) as a displacement of the membership curve along the abcissa axis; and that this prediction gives better agreement with the experimental result for VERY than the 'traditional' fuzzy set mu-square operation.

As far as is known to me, there exist no formal semantic experiments in connection with the LB-YN assumption except, perhaps, one result of the experiments of Hersh and Caramazza (1976) already mentioned in sect. 1. Namely the inconsistency in the answers given by their subject 4, compared with the answers of the other subjects, concerning the labels 'large' and 'small'. We believe that this inconsistency is due to the fact that it was not made clear to the subjects i) whether their answers concerning 'large' and 'small' should refer to a label set which contained the elements 'VERY large' and 'VERY small' also; ii) it was not made clear whether the experiment referred to a YN or an LB situation. Fig. 4 shows that according to the LB-YN assumption of the TEE model, the membership curve for 'tall' is bell-shaped for LB reference and S-shaped for YN reference when the reference label set contains the element 'VERY tall' also. It is thus not surprising that one observes intersubject inconsistencies when the reference situation and label set are not made clear to the subjects. In such a situation, each subject is forced to make his own choice of a reference situation and label set, and we have no guarantee that different subjects will make the same choice.

Definition 15. The LB-YN assumption. This assumptions, which concerns the relation between the quantization intervals of LB and YN experiments, consists of many parts which require precise definitions of the situations to which they refer. For the convenience of the reader, we therefore state here only an informal summary of the main points of the assumption. The formal definition is given in the appendix A2.

Informal statement of the LB-YN assumption: The quantization interval of a given label b, as used by a single subject S, is the same in an LB and a YN experiment, except when the reference label set also contains a label a whose quantization interval Δu_a is not disjoint from that of b in a YN experiment. When Δu_a is a subset of Δu_b in a YN experiment in such a way that either the upper or the lower bounds of the two quantization intervals coincide (e.g. when a=VERY b), then the quantization intervals of a and b *in an LB experiment* are disjoint. The quantization interval of b in a YN experiment is the union of the quantization intervals of a and b in the LB experiment. (For example, the quantization interval of b=tall in an LB experiment is the difference between the quantization intervals of b=tall and of a=VERY tall in a YN experiment, see fig. 4.)

Before we state the assumption for the VERY modifier, we need a definition of extremal concepts and labels.

Definition 16 of extremal and nonextremal concepts and labels. Let the label λ represent the linguistic value of a one-dimensional attribute. λ, as well as the concept for which it is a label, are called upper extremal iff the upper bound of the quantization interval Δu_λ (in connection with a YN or LB experiment) coincides with the upper bound of the universe U. An analogous definition holds for a lower extremal concept.

Examples of upper extremal concepts referring to YN experiments are 'tall', 'old' and their qualifications by 'VERY' and 'VERY VERY' etc. . 'small', 'young' and their VERY-qualifications are examples of lower extremal concepts. 'medium' is a nonextremal concept. Note that a label which is extremal for a YN experiment, is not necessarily extremal for an LB experiment (see def. 15).

When extremal labels exist for a given attribute, then it is not usual to use a VERY qualification (such as 'VERY medium') for those labels which are nonextremal in a YN situation.

A label is extremal in an N-dimensional universe if, for given values of N-1 attributes, the upper or lower threshold of the last attribute coincides with the upper or lower threshold of the universe respectively. 'Obese' and 'slim' (see fig. 3), as well as their qualifications by VERY, are examples of extremal labels in a 2-dimensional universe.

'Hue', or 'colour' in everyday terminology, is an example of an attribute for which no extremal labels exist. In the chromatic diagram (Boynton, 1984), all the different hues (for a given saturation) lie on a closed, horseshoe-formed curve. Thus there is no upper or lower bound of the universe because every colour has a neighbour. E.g., 'yellow' has the neighbour 'green' on one side and 'red' on the other. 'Red' has the neighbours 'yellow' and 'purple' (assuming that we operate with the label set $\Lambda = \{\text{blue, green, yellow, red, purple}\}$. In this case *each* of the labels may be qualified by 'VERY'.

end def. 16

Definition 17 of the assumption for the VERY modifier. Consider two YN experiments performed by the same subject, and referring to the same label set Λ which contains both the label 'λ' and the label 'VERY λ'. The specified label (see def. 3) of the first experiment is $\lambda_{spec} = \lambda$, and that of the second experiment $\lambda_{spec} =$ VERY λ. Then the assumption says that the quantization interval for 'VERY λ' is a subset of the quantization interval for λ. The location of the subset depends on the type of attribute A to which λ refers.

When 'λ' is an upper (lower) extremal concept (see def. 16), then 'VERY λ' is also an upper (lower) extremal concept. When A is an attribute such as 'hue', for which the quantization intervals of the different labels lie around a closed curve (see def. 16), then the quantization interval of 'VERY λ' is obtained by a narrowing-down on each side of that for 'λ'.

Thus modification by VERY is a means by which the characteristics of the concept λ are accentuated.

end def. 17

Fig. 4 shows the nonfuzzy, step-shaped $P(\lambda|u)$ threshold curves for λ =tall (full curve) and λ =VERY tall (broken curve) respectively according to def. 17 and the first two assumptions of the TEE model; assuming a subject whose lower thresholds for these two labels are 170 and 180 cm respectively. In addition the figure shows the nonfuzzy, square-pulse shaped threshold curve for λ =tall with LB reference (dotted curve) according to the LB-YN assumption of def. 15.

Assuming a $P(u_j|u_i^{ex})$ error curve such that S's estimate u of the attribute falls into the i-th (i.e., the correct) quantization interval in 50% of all cases and into the next lower and the next higher quantization interval in 25% of all cases, it can then be shown that these curves are rounded off to the $P(\lambda|u^{ex})$ S- and bell-shaped likelihood curves of fig. 4 for which the abcissa axis is u^{ex} instead of u. Furthermore, according to the third, or LB,YN-MU assumption of the

TEE model (Hisdal 1988a), these rounded-off threshold curves are equal to S's membership curves as elicited in an exact MU experiment; assuming that the real $P(u|u^{ex})$ is equal to $P^{est}(u|u^{ex})$, S's estimate of the error curve under the conditions to which he refers his fuzziness #1; and that S is an ideal subject (Hisdal, 1988a). Observe that the membership curve for 'tall' with LB reference is subnormal, its greatest membership value is smaller than 1. This is due to the fact that the size of our quantization interval for LB-tall is smaller than the width of the error curve (see consequence 7 in Hisdal 1988a).

Zadeh (1973) has suggested that the membership values for 'VERY λ' are equal to the square of those for 'λ'. The membership curve for 'VERY tall' according to this formula, as derived from that for 'tall', is shown by the dotted curve in fig. 5.

Both Hersh and Caramazza (1976, p. 265) and Norwich and Turksen (1984, fig. 3b) indicate that a displacement of the membership curve for λ along the u^{ex} axis is a better representation of 'VERY λ' than the squaring operation.

In the TEE model, a displacement operation for the VERY modifier (assuming YN reference of the unmodified label) holds exactly according to def. 17 and the LB,YN-MU assumption of equivalence when the estimated error curve is u^{ex}-invariant over the pertinent interval of u^{ex} values. This is shown by the YN membership curves of fig. 4 which are drawn again in fig. 5, full curves. The displacement law holds also, according to the TEE model, (for the expected $\mu_\lambda(u^{ex})$ curves) when the membership curves are elicited under nonexact conditions (assuming a u^{ex}-invariant real error curve). In that case the full curves are rounded off once more by the real $P(u|u^{ex})$ error curve to the broken curves of fig. 5 (see Hisdal, 1986b, fuzziness #1b).

7. Postscript and Summary

In this paper we have tried to keep a reasonable balance between the description of the actual, situation-dependent interpretations of labels as used in natural language communication and information exchange; and the description of a situation-dependent theory which can explain these interpretations in a consistent way, starting from the subject's observational data.

In many ways this paper has been the most difficult one to write of the TEE model series. This is probably due to the fact that the traditional exact sciences, which operate with absolute numbers and units, have not developed good tools and terminology for taking context- and situation-dependence into account.

On the other side of the extreme we have the terminology of 'modes' in computer science. On my pocket calculator, the period has the meaning of a decimal point when the calculator is in the 'calculate' mode. When it is in the mode for time setting, the same sign (period) has the meaning of 'p.m.'.

Human language has succeeded in taking an intermediate position between these two extremes. An adjective label such as 'big' can refer to completely different numerical values, depending on the noun to which it is explicitly or implicitly attached. However, if we convert the numerical values to relative ones by dividing them by the size of the u interval in the given context (e.g., the interval for all possible sizes of dogs), then the meaning of 'big' is, on the whole, the same for all nouns. Namely a size value lying in the upper part of the interval applying to all objects named by the particular noun. A 'big dog' is 'big' in relation to all dogs, and a 'big house' is 'big' in relation to all houses in a particular region. Thus adjective labels signify, for the most part, *relative* numerical values. However, once this relativity is taken into account, the meaning has a common denominator, independent of the particular noun.

We shall here adopt the terminology of Barwise and Perry (1983) whose book treats the situation dependence of natural language, although not in connection with adjective labels. They talk about the basic *meaning* of a term, as contrasted with its different *interpretations*. The interpretation of a term is derived from its basic meaning and the particular situation in which it is used.

Assuming that the noun to which the adjective label λ is attached is known, we have seen that a label λ such as 'big' or 'tall' can have different interpretations, depending on whether it is used in an LB or a YN situation, and on whether it refers to a label set which contains the label 'VERY λ' as well. However, the two resulting interpretations of 'tall' are built on a single meaning of 'tall' and 'VERY tall', namely the subject's quantization interval in U for 'Y-tall' and 'Y-VERY tall' in a YN experiment. A similar YN versus LB interpretation-modifying effect occurs for a label such as 'tall ORA medium'.

Still another situation are MU experiments. The subjective membership values for λ elicited in such experiments depend on all the above mentioned reference situation; namely the noun phrase reference (e.g. 'adult female human' for λ=tall), the reference label set and, in the above mentioned cases, on the LB versus YN reference. The last reference situation can completely change the shape of the membership funtion (S-shape for Y-tall versus bell-shape for LB-tall when the reference label set contains 'VERY tall').

A really intelligent computer system which simulates natural language should have the ability to represent the single basic meaning of a label which is independent of the noun attachment, of the YN versus LB reference, and of the inclusion, or non-inclusion, of 'VERY λ' in the reference label set; and to derive the different interpretations of this label, depending on these three references. Hersh and Caramazza's experiments, in which subjects had to attach labels such as 'large' or 'small', or membership values concerning such labels, to squares of different sizes are particularly interesting in this connection. It seems extremely improbable that human beings permanently store membership values, or absolute quantization intervals, for large and small squares. However, Hersh and Caramazza's subjects were shown in advance the range of squares of different sizes which were going to be used in their experiments. This prior information enabled them to answer questions concerning the approppriateness of a particular label to a particular square. The knowledge concerning the basic meaning of labels such as 'large' and 'small' does not, of course, prevent a subject or computer system from storing permanently the noun dependent interpretations of these labels, e.g. the YN quantization intervals and memberhip functions of 'tall woman', 'tall man', tall 'giraffe'.

The assumption of the existence of a basic meaning of a term, as compared with its different, situation-dependent interpretations, is shown in appendix A1 to be especially relevant in connection with the negation. The basic meaning of 'NOT λ' is that of a label attached to objects belonging to the traditional complement of the subset of objects labeled λ by the subject. In sect. 5 it was then shown that the fuzzy set 'one-minus' postulate for the negation can be *derived* from this basic nonfuzzy meaning. Other derived results in this paper are the 'summing up to one' postulate of fuzzy clustering algorithms (equation (28)), as well as the displacement result betwen the membership curve of a label and its 'VERY' modification (sect. 6 and figs. 4, 5), and between the membership curve of the basic negation of a label and its antonym (appendix A1 and fig. 6).

All these results depend on the very simple first and second assumptions of the TEE model (sect. 4) according to which the answers given by the subject in LB and YN (but not in MU) experiments are determined by the nonfuzzy belongingness, or non-belongingness, of the object's estimated attribute value u to the subject's nonfuzzy quantization interval for the label λ in the universe U .

The definition of label sets which are complete and nonredundant in LB situations (def. 1, item 9 and def. 14) seems, at first sight, to contradict the basic tenet of fuzzy set theory; namely the possibility that a subject will assign a partial membership value to an object in a class. However, by establishing a clear differentiation between LB or YN versus MU situations (sect. 3), we show in Hisdal (1988a) that the assignment of a partial membership value in a MU situation does not contradict the nonredundancy of a label set in an LB or YN situation, assuming (1) That a subject bases his answers in an LB or YN situation on nonfuzzy thresholds in the universe of *estimated* attribute values (defs. 10, 11); and (2) That a subject estimates the resultant spread of answer values among objects with a given exact attribute value u^{ex} (Hisdal 1988a, LB,YN-MU assumption). This description refers to fuzziness #1. Analogous statements hold for fuzziness #2 or 3 (Hisdal 1986b).

Throughout this paper we have noted the difficulty of performing meaningful, formal semantic experiments, such that the subject can identify the real-life situation to which the experiment is meant to refer. And we have suggested (see end of appendix A1) that the analysis of recorded, real-life conversations, or of sentences in the literature (including their context-embedment) may be an important supplementary experimental method.

Appendix A1. More on Negation and Antonyms

The purpose of this appendix is 1) To provide a more formal definition of the negation of sect. 5, based on the results of YN experiments and resulting in the 'one-minus' law for grades of membership. We shall consider this definition to be the basic 'interpretation I' of the negation. 2) To discuss the degree of agreement between the 'one-minus' formula and experimental results. 3) To discuss the relation between negation and antonyms according to the TEE model and compare it with experimental results. 4) To point out the existence of two higher level interpretations IIa, IIb of the negation used for purposes of politeness. These interpretations are built on top of interpretation I in the sense that the subject is aware of interpretation I and makes use of it in the definition of the higher level interpretation. 5) To discuss interpretation IIc of 'NOT VERY λ', in the sense of '(NOT VERY) λ', also built on top of interpreation I. 6) To point out the difficulties of formal experiments concerning the negation due to all these different interpretations in a natural language situation.

Our conclusion is that although we must be aware of the existence and use of higher level interpretations of the negation in natural language, all of these interpretations are built on the single, basic interpretation I in which the quantization interval for 'NOT λ' is the complement of the quantization interval for λ with respect to U; resulting in the 'one-minus' law for grades of membership. This conclusion agrees, on the whole, with that reached by Bandler & Kohout (1985, p. 768) when referring to the work of Trillas (1979) and of Esteva & Domingo (1980). Bandler & Kohout say: "Most agree in setting the value $\overline{a}$ of *not-A* as $1-a$. Trillas and his colleagues have shown that there is not much generality to be gained by varying this formula for negation, and we will adopt it in this paper".

When it comes to antonyms, we conclude that both according to experimental results, and according to the TEE model, there is a greater contrast between a label and its antonym (e.g. 'young' and 'old') than between a label and its negation ('young' and 'NOT young'); and that this result does not quite agree with the formulation of linguists concerning the difference between an antonym of a label and its negation.

We have already shown in sect. 5 that Zadeh's postulated 'one-minus' formula for the negation can be derived in the TEE model, assuming YN reference of the MU experiment. In this most basic, and logically most straightforward interpretation I, the subject processes the question (or the equivalent statement) 'Is John NOT λ?' by replacing 'NOT λ' by a label 'λ''=NOT λ to obtain the question 'Is John λ'?'. A 'N' answer to this question is now equivalent to a 'Y' answer to the question 'Is John λ?' Thus double negation is equivalent to affirmation in interpretation I, in agreement with the 'one-minus' law which gives $\mu_{NOT\ (NOT\ \lambda)} = 1 - (1 - \mu_\lambda) = \mu_\lambda$. The following is a formal definition of interpretation I.

Definition A1. The basic interpretation I of negated attribute values in the TEE model. In this interpretation, the subject's quantization interval in U for '*NOT* λ' is the traditional complement with respect to U of the quantization interval of '*Y*-λ' in a YN-experiment; or, equivalently, it is the quantization interval of '*N*-λ' in a YN experiment. The membership curve of '*NOT* λ' with respect to U^{ex} is then found in the usual way from $\Delta u_{NOT\ \lambda}$, making use of $P^{est}(u|u^{ex})$, the subject's estimate of the error curve in connection with his fuzziness #1 (see Hisdal, 1988a). The fuzzy set 'one-minus' law for the negation,

$$\mu_{NOT\ \lambda}(u^{ex}) = 1 - \mu_\lambda(u^{ex}) \qquad (A1)$$

follows from this interpretation. (For the proof, see section 5).

end def. A1

Definition A1 presupposes a negated specified label in a YN or YN-MU experiment. Although most subjects are well aware of the interpretation I resulting from this definition, (see the experimental results and the theoretical considerations below), there is not much point in using negated adjective labels in this interpretation. For example, it is easier to answer the question 'Is John tall?' by a 'Y' answer than to answer 'N' to the question 'Is John NOT tall?'. In an LB situation, the amount of information conveyed by either 'John is small', or 'John is of medium height', is bigger than the amount of information conveyed by 'John is NOT tall'. The English language, and probably many other ones, have therefore developed other interpretations of negated labels, assuming that people will seldom use interpretation I, and are able to identify the intended interpretation from the context. Three such higher level interpretations are presented below.

Because of this ambiguity of negated labels we must expect difficulties with formal experiments concerning the negation unless the experimenter takes great pain to make it clear to the subject to what natural language discourse situation he is to refer. Even then it is possible that many subjects will mostly make use of interpretation I due to the formal experimental situation in which they find themselves.

The general overall shape of the experimentally found membership curves for '*NOT* λ' versus those for 'λ' agrees with the 'one-minus' law according to Hersh & Caramazza (1976) and Norwich & Turksen (1982b, 1984). However, there are deviations between the 'one-minus' law and the exact numerical values found by the experimenters. Furthermore, the deviations from the 'one-minus' law found by the two groups of experimenters are of opposite signs. Thus Hersh and Caramazza (see, e.g., their figs. 2, 3) find practically consistently that

$$\mu_{NOT\ \lambda}(u^{ex}) > 1 - \mu_{\lambda}(u^{ex}) \qquad (A2)$$

except for those u^{ex} values for which μ_{λ} is small. For the latter there is good agreement with the 'one-minus' law. Norwich & Turksen (1982b, figs. 1,2; 1984, fig. 3b) show only the membership curves for two subjects. For one of these there is a good agreement with the 'one-minus' law, for the other there is a very significant deviation with the opposite sign to that of equation (A2).

In contrast to the disagreement between the two groups of experimenters concerning the deviations from the 'one-minus' law, they are in complete agreement concerning the relation between the membership curve for 'NOT λ' and that for the antonym of λ. Thus Hersh & Caramazza (1976, p. 265) find that 'NOT large' appears to extend the concept 'small' to include the midrange of the continuum". A similar result is found by Norwich & Turksen (1984, p. 13).

The latter results of Hersh & Caramazza's and Norwich & Turksen are easily explained in the TEE model by assuming that subjects usually refer to a label set containing a label such as 'medium' (see, e.g. equation (1)). The quantization interval of 'N-tall' in a YN experiment is then the union of the quantization intervals of 'Y-small' and 'Y-medium'. The resulting membership curves for 'small' (full curves) and for 'N-tall' (broken curves) are shown in fig. 6. We see that the 'small' curve corresponds to a displacement of the 'N-tall' curve to the left by an amount equal to the quantization interval of 'medium'. Only for a twin reference label set (def. 6) do we obtain semantic equivalence between the negation and the antonym of a label. The experimentally found displacement between the two membership curves shows that the natural reference label set used by subjects contains also an intermediate label such as 'medium' (see also last paragraph of section 3 in this connection).

The linguists Quirk & Greenbaum (1979, p. 431) say that the prefix 'un', which is used to form antonyms, expresses "the opposite end of a scale". While the negation prefix 'non' "expresses non-gradable binary contrast". This formulation may give the impression that there is a greater contrast between a label and its negation than between a label and its antonym; a result that agrees neither with experiment nor with the TEE model according to both of which the contrast between a label and its antonym is the greater one. The meaning of a twin label (such as 'young' or 'old') is not changed by being contrasted with its negation, versus being contrasted with its antonym. 'Binary contrast' and 'negation' simply refer to the situation of a YN-experiment concerning the single label $\lambda_{spec} = \lambda \in \Lambda$. While λ and its antonym refer to an LB-experiment; or to two YN-experiments referring to the same label set Λ, using $\lambda_{spec} = \lambda$ and $\lambda_{spec} = \text{antonym}(\lambda)$ respectively. All these experiments result in a contrast between λ and its antonym, and between λ and $N\text{-}\lambda$, the former being greater than the latter. The contrast is nonfuzzy (or a 'binary contrast' in linguistic terminology) when referred to U, and fuzzy when referred to U^{ex}; 'fuzziness' meaning that objects with the same u^{ex}-value may be assigned different answer values. By taking this situation into account, a subject can assign membership values (or graded values in the linguistic terminology) both to λ and its antonym, as well as to its negation, according to the LB,YN-MU assumption of equivalence (Hisdal, 1988a).

A very fine nuance of the difference between the antonym of λ and its negation is often used for reasons of politeness in natural language. We know that it is not *comme il faut* to say 'X is an unpleasant person'. In order to soften this formulation, a polite person will often say 'X is NOT a pleasant person', because the basic interpretation of 'NOT pleasant' includes also the medium region of pleasantness. Thus we see that natural language makes use of two interpretations of 'NOT pleasant': The basic interpretation I which conforms to the 'one minus' law; and the higher level antonym interpretation IIa. The speaker is well aware of the basic interpretation I which helps to make his statement more polite. And the listener is well aware of the trick used by the speaker. (In the above description, 'pleasant' and 'unpleasant' may be replaced by 'VERY pleasant' and 'VERY unpleasant'.)

The following definition sums up the situation.

Definition A2. The antonym interpretation IIa of the negation. Let λ^1 be an extremal label, and let λ^2 be its antonym. In an LB situation, a subject may assign, for reasons of politeness, the label '$\underline{\text{NOT}}$ λ^1' or '$\underline{\text{NOT}}$ VERY λ^1' (with the accent on the word 'NOT') to those objects to which the label 'VERY λ^2' or 'λ^2' actually applies. The negated label thus replaces the more restrictive, and therefore more correct, antonym label. The latter expresses the real meaning which the subject has in mind, but is reluctant to use openly.

Another higher level interpretation of the negation, probably also connected with politeness, occurs in YN situations using a negated specified label. When the questioner asks

$$\text{Isn't John tall?} \tag{A3}$$

then S may answer 'yes, he sure is'; meaning 'John is tall', not 'John is NOT tall'. This interpretation of 'NOT tall' in (A3), resulting in 'Y' answers for large height values when the 'logical' answer according to interpretation I should be 'N', is a possible explanation of the results of Hersh and Caramazza, equation (A2) here, according to which a minority of subjects answered 'Y' concerning the specified label 'NOT large' for big sizes of their squares.

Vice versa, S will answer 'No he isn't' to (A3) when he means that John indeed is 'NOT tall'. At the same time, many subjects are aware of the fact that their answer is actually 'wrong'; i.e., they consider def. A1 to be the basic interpretation of the negation.

Seen from the point of view of the basic interpretation I, S thus actually answers in all cases the affirmed question

$$\text{Is John tall?}\ , \tag{A4}$$

not the question (A3).

Definition A3 of the affirmed interpretation IIb of negated labels. In this interpretation a negated specified label 'NOT λ', or some colloquial equivalent, is used in a YN question. The subject then replaces this label by the affirmed label 'λ' in the YN question, and answers the resultant question according to the basic interpretation I of the negation, def. A1.

I believe that at least part of the reason for this seemingly illogical behaviour of the subject is that many people consider a negative answer to be impolite. The formulation (A3) of the question, with its implied 'wrong' answers of interpretation IIb, gives S the opportunity to be polite, whatever his opinion is. A 'Y' answer is always polite. And a 'N' answer to (A3) is also polite because it is a repetition of the negation already contained in the question (A3), thus seemingly indicating agreement with the sentence of the questioner when its interrogative form is converted to a statement.

A negated interrogative form of an assertive statement which the speaker considers to be an obvious truth is also used extensively for oratorical purposes.

$$\text{We have not done enough for the country?} \qquad (A5)$$

says the foreign-origin, American father who has lost his son in the war and now hears that his nephew is going to join the armed services (Potok, 1983).

Intuitively, we all understand that what he wants to say is: "We have done more than enough for the country". Why then does he use an interrogative sentence, and why in the negated form?

The interrogative form is again a superficial form of politeness which pretends to leave it open to the listener to express his own opinion. And this would indeed have been the case if the sentence had been the straightforward affirmed one,

$$\text{Have we done enough for the country?}, \qquad (A6)$$

Both a Y and a N answer would then be acceptable from the speaker's point of view.

The seemingly unnecessary negated form in (A5) is a signal to the listener that the speaker does not want him to have such a freedom of answer. He wants to force his opinion upon the listener, daring him to answer N to the affirmed question (A6). The listener can now give a polite "You sure have!" answer to (A5), where it is again understood by both parties that he interprets the question in the sense of (A6).

Thus interpretation IIb of def. A3 serves two purposes. a) As we explained in connection with sentences (A3), (A4), it gives the listener the opportunity of an answer which seemingly agrees with the declarative form of the sentence uttered by the speaker, no matter whether he agrees with the speaker or not.

b) It may also be used by the speaker as in the case of (A5) for the purpose of forcing his opinion upon the listener. The speaker believes that, even to the meanest intelligence, it must be clear that the declarative form of the negated question (A5) is false. He thus implicitly uses an accepted method of proof in mathematics; namely 'proving' the falseness of the negation of a theorem.

Finally we mention still another interpretation of the negation which applies only to the prefix 'NOT' in front of 'VERY'.

Definition A4 of interpretation IIc of

$$\text{NOT } \underline{\text{VERY}}\ \lambda\ , \qquad (\text{John is NOT } \underline{\text{VERY}} \text{ tall}) \qquad (A7)$$

used in an LB situation (which may replace a direct 'N' answer to the YN question 'Is John VERY tall?'). The accent in (A7) is on the word 'VERY'. In this case the word 'NOT' modifies the modifier 'VERY', not the complete phrase 'VERY λ'; resulting in an intended meaning of 'λ BUT NOT VERY λ' (tall BUT NOT VERY tall). The interpretation of the label in (A7) is then the same as that of 'λ' in an LB situation referring to the quintuple label set (15) (see curve for 'LB-tall' in fig. 4).

We have thus at least four possible interpretations of the negation, I, IIa, IIb, IIc. In addition it may happen that a subject without aptitude for mathematics tries to use the basic interpretation I, but gets confused in connection with double negation. It is not a trivial matter to understand that $(-1)^2 = 1$. We must therefore expect inherent ambiguities in every formal experiment concerning the negation.

The best way to perform an experiment is probably to search for negations in audio recordings of everyday conversations, and in stories in the literature; and to elicit the intended meanings of the negated terms from subjects who are presented with these recordings or stories.

Because of the ambiguities connected with the negation, we have not explicitly included negated labels as elements of legal label sets in this paper. Nor do we recommend negated specified labels for YN situations. This is in contrast to Hisdal (1984a,b) in which we explicitly defined label sets such as $\{\text{tall, NOT tall}\}$, or $\{\text{tall, medium, NOT (tall or medium)}\}$ to be legal ones, although requiring reference to a label set without negated labels.

Appendix A2. Formal Definition of the LB-YN and the YN-YN Assumptions

In section 6, def. 15, we presented an informal definition of the LB-YN assumption. The precise definition of this, as well as of some related assumptions, takes up quite a bit of space and we have therefore deferred it to this appendix. The assumptions are based on informal semantic experiments concerning the interpretation of various labels in different situations.

In all the definitions of this appendix we assume that a single subject S performs one or more semantic experiments, exp1, exp2, ... referring to the same attribute (e.g. 'height'), and the same context dependent class of objects *OB* (e.g. 'women'). The quantization interval for a label λ which S uses in expj will be denoted by $\Delta u j_{\lambda}$.

The different experiments to which the definitions of this appendix refer are listed in fig. 7. The reference label set of the first two of these,

$$\Lambda = \{\lambda_l\} , \qquad l = 1, \ldots, L \tag{A8}$$

is assumed to be legal and YN-nonredundant (see def. 1, item 9), and to include the label

$$\lambda_{spec} = b \in \Lambda . \tag{A9}$$

As an example, Λ may be the set

$$\Lambda = \{\text{small, medium, tall}\} . \tag{A10}$$

Definition A5 of the LB-YN assumption for YN-nonredundant label sets. This definition refers to the LB exp1 and the YN exp2 of fig. 7, both of which refer to the same, YN-nonredundant label set Λ. The specified label b of the YN experiment (see def. 3) can be any element λ_l of Λ. The assumption then says that the quantization intervals for b in exp1 and exp2 respectively are identical,

$$\Delta u1_b = \Delta u2_{Y-b} \qquad \forall\, b = \lambda_l, \quad l = 1, \ldots, L . \tag{A11}$$

For example, the quantization intervals for each of the three labels of (A10) are the same whether the experiment is an LB or YN one.

end def. A5

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Experiments 3, 4, 5 of fig. 7 refer to a label set Λ' defined as follows. Λ' consists of the elements of Λ, plus an additional element a which will be defined in more detail below,

$$\Lambda' = \Lambda \cup \{a\} \ . \tag{A12}$$

Definition A6 of the YN-YN assumption for a YN-redundant label set. Consider the YN-experiments 3 and 4 in fig. 7 both of which refer to the same YN-redundant label set Λ', equation (A12), but to different specified labels b and a respectively. The assumption of def. A6 then says that for some $b \in \Lambda$ there exists a label a such that the following three assertions hold for experiments 3 and 4, assuming a one-dimensional attribute universe.

1. The quantization interval of Y-a in exp4 is a subset of that of Y-b in exp3,

$$\Delta u4_{Y-a} \subset \Delta u3_{Y-b} \ . \tag{A13}$$

2. Either the upper or the lower bounds of the two quantization intervals coincide; i.e.,

$$\text{either} \qquad u4_{(Y-a),u} = u3_{(Y-b),u} \tag{A14}$$

$$\text{or} \qquad u4_{(Y-a),l} = u3_{(Y-b),l} \tag{A15}$$

3. When equations (A13) and (A14) or (A15) hold, then the label set Λ' is a legal, though YN-redundant one.
   For example, the label set

$$\Lambda' = \{\text{small, medium, tall, VERY tall}\} \ , \tag{A16}$$

is legal, though YN-redundant. The quantization interval of 'Y-VERY tall' is a subset of that for 'Y-tall' such that equation (A14) holds.

Definition A7 of the YN-YN assumption concerning the label b. Consider the YN experiments 2 and 3 in fig. 7, referring to the YN-nonredundant and the YN-redundant label sets Λ and Λ' respectively. The assumption then says that the quantization interval for b is the same in both experiments,

$$\Delta u2_{Y-b} = \Delta u3_{Y-b} \ . \tag{A17}$$

For example, the quantization interval for 'Y-tall' is the same whether the YN experiment refers to the label set (A10), or to (A16).

Definition A8 of the assumption of legality of the LB experiment 5. The assumption says that the LB experiment 5, which refers to the YN-redundant label set Λ', is a legal one. It follows that Λ' is a nonredundant label set ('nonredundant' means 'nonredundant in the context of an LB experiment', see def. 1, item 9), and consequently that

$$\Delta u5_b \cap \Delta u5_a = \emptyset \ . \tag{A18}$$

For example, the quantization intervals of 'tall' and of 'VERY tall' are disjoint in an LB experiment.

Observe that the expression 'quantization intervals' (without qualifications) always refers to the universe U of *estimated* attribute values. Disjointness of these intervals (in a YN or LB experiment) does not necessarily imply disjointness of the quantization intervals in U^{ex}. It is the recognition of this fact by the subject which gives rise to the grade of membership concept.

Definition A9 of the assumption of inequality of the quantization interval of b in a YN and an LB experiment respectively both of which refer to Λ'. Consider experiments 3 and 5 of fig. 7. The assumption says that

$$\Delta u5_b \cup \Delta u5_a = \Delta u3_{Y-b} \ . \tag{A19}$$

From equations (A18), (A19) it then follows that

$$\Delta u5_b = \Delta u3_{Y-b} - \Delta u5_a \subset \Delta u3_{Y-b} \ . \tag{A20}$$

For example, the quantization interval of 'tall' in an LB experiment is a subset of the quantization interval of 'Y-tall' in a YN experiment, although both experiments refer to the same label set Λ'. Furthermore, the union of the quantization intervals for 'tall' and 'VERY tall' in the LB experiment 5 is equal to the quantization interval of 'Y-tall' in the YN experiment 3 whose reference label set does not contain 'VERY tall'.

Definition A10 of the assumption of equality of the quantization intervals of a in a YN and an LB experiment respectively. Consider experiments 4 and 5 of fig. 7. The assumption says that

$$\Delta u5_a = \Delta u4_{Y-a} \ . \tag{A21}$$

For example, the quantization interval of 'VERY tall' is the same in the LB experiment 5 as in the YN experiment 4.

Definition A11 of the BUT NOT assumption. Consider a YN experiment referring to Λ'. Then the assumption says that the specified label $c = b$ BUT NOT a is a legal one. Furthermore the assumption says that when the label b is replaced by c in Λ', then the resulting label set Λ'' is still legal, and is in addition YN-nonredundant.

The BUT connective is used instead of AND in those cases in which there is a contrast between the components which it connects (in our case an affirmed versus a negated label). It then follows from defs. A11, A10 that the quantization interval of Y-c is equal to that of b in exp5. And from def. A5 it then follows that the quantization interval of c in an LB experiment referring to Λ'' is equal to that of b in an LB experiment referring to Λ'.

For example, the quantization interval of 'tall BUT NOT VERY tall' in a YN experiment is equal to that of 'tall' in an LB experiment, assuming that both refer to (A16). And it is also equal to the quantization interval of 'tall BUT NOT VERY tall' in an LB or YN experiment referring to the label set of (A16) with the element 'tall' replaced by 'tall BUT NOT VERY tall'.

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Δu_λ	$= \{u_{\lambda l}, \ldots, u_{\lambda u}\}$ = quantization-interval for the label λ. Nonfuzzy interval in U for which the subject assigns the label λ in an LB (labeling) or YN (yes-no) experiment (definition 11).
est	superscript for the estimate by the subject of a probability distribution relating to his fuzziness #1a, 2a or 3a (Hisdal 1986b).
est-nexcond	superscript for a quantity elicited under real conditions of observation which are identical with the estimated ones assumed by the subject in connection with his fuzziness #1a.
Exact	conditions or experiment. A semantic experiment in which the subject measures (or is told) the exact attribute value of each object. Consequently $u = u^{ex}$ for an exact experiment.
excond	superscript for a quantity elicited under exact conditions of observation for which $u = u^{ex}$.
Fuzziness #1a	is due to the subject's awareness of the possibility of errors of estimation of the attribute value (Hisdal 1986b, 1988a).
Fuzziness #3a	is due to the subject's awareness of the possibility of variations between different persons of the thresholds u_λ for the label λ (Hisdal 1986b).
λ	a label; e.g. 'tall', or 'VERY tall', or 'tall OR medium'. The same symbol is used for the concept itself (see Kohout 1988 for the differentiation between these two). In fuzzy set theory it is usual to identify this concept with its membership function $\mu_\lambda(u^{ex})$, and to call it the "fuzzy set λ". We use a lower case letter to denote this concept and its label instead of the more usual A or F of fuzzy set theory, because we need the corresponding upper case letter for a label set.
$\Lambda = \{\lambda_l\}$	$l = 1, \ldots, L.$ A label set; e.g., ' {small, medium, tall} ' (def. 1, item 9).
LB experiment	a Labeling experiment in which a subject assigns a label from a label-set to an object (definition 2).
LB,YN-MU	assumption of equivalence of the TEE model. The assumption that the grade of membership value is a means by which a subject takes account of the existence of fuzziness in everyday life. He interprets the μ_λ value which he assigns to an object of attribute value u^{ex} (in a MU experiment performed under exact conditions) as the proportion of objects with that value of u^{ex} which he would label λ in an LB or YN experiment under the conditions of observation to which he refers his fuzziness #1a: $\mu_\lambda^{excond}(u^{ex}) = P^{est-nexcond}(\lambda \mid u^{ex}) = \sum_{u=-\infty}^{\infty} t_\lambda(u)\ P^{est}(u \mid u^{ex})$. For fuzziness #3a, the subject interprets μ_λ as the proportion of subjects who would label the object λ in an LB or YN experiment (Hisdal 1986b, 1988a).
μ_λ	membership value in class λ assigned by the subject to a given object in a MU experiment under given conditions of observation. μ_λ is a unique function of u, but not of u^{ex} according to the TEE model; assuming a given reference label set Λ, and a given reference to either a YN or an LB situation (see definitions 4, 5).

Fig. 1a. Notation and Terminology (continued in fig. 1b).

Note that many of the main symbols are defined in section 2, def. 1, items 1-11. These definitions are mostly not repeated here. They concern the experimenter E (item 1), the object set OB (item 2), the subject S (item 3), the answer value (item 4), the uni- or multi-dimensional attribute A (item 5), the exact experimenter experiment or the E-experiment (item 6), the I subsets OB_i of OB and their cardinality $card_i$ (item 7), the unqualified probability distribution over U^{ex} (item 8), a legal (complete, nonredundant) reference label set Λ (item 9), an exact semantic experiment (item 10), the set of conditions of observation, and constant and exact conditions of observation (item 11).

$\mu_\lambda^{nexcond}(u^{ex})$	Under nonexact conditions, μ_λ is a unique function of u, but not of u^{ex}. We therefore define its expectation over all objects with attribute value u^{ex}, Exp $\{\mu_\lambda^{nexcond}(u^{ex})\} = \sum_{u=-\infty}^{\infty} P(u\|u^{ex})\ \mu_\lambda^{excond}(u)$. Thus $\mu_\lambda^{excond}(u^{ex})$ is a rounded version of the nonfuzzy threshold curve $t_\lambda(u)$; and Exp $\{\mu_\lambda^{nexcond}(u^{ex})\}$ is a rounded version of $\mu_\lambda^{excond}(u^{ex})$. The rounding-off being performed by a convolution with $P^{est}(u\|u^{ex})$ and $P(u\|u^{ex})$ respectively.
MU experiment	a grade of membership experiment in which the subject is asked to assign a grade of membership value $\mu \in \{0, \ldots, 1\}$ to an object concerning the label λ (definitions 4, 5).
nexcond	superscript for a quantity elicited under nonexact conditions of obervation. Note that u^{ex} (the exact attribute value of the objects as measured by the *experimenter*) may be an argument such a quantity.
ORA, ORE	inclusive (OR/AND) and exclusive OR connective respectively.
$P(\lambda\|u^{ex})$	labeling probability or likelihood distribution of λ over u^{ex}. Probability that an object with attribute value u^{ex} will be labeled λ in a YN or LB experiment. Superficially stated, it is later identified with $\mu_\lambda(u^{ex})$ elicited in a MU experiment (see LB,YN-MU assumption of equivalence).
$P(u^{ex})$	unqualified or prior probability distribution over u^{ex}; e.g., the distribution over height of the population of objects, unqualified by the label λ.
$P(u\|u^{ex})$	real error curve for a given subject, and a given set of conditions of observation. When $P(x)$, the probability of an error $x = u - u^{ex}$, is independent of u^{ex}, then we talk about a 'u^{ex}-invariant' error curve.
$P^{est}(u\|u^{ex})$	estimated error curve; the subject's estimate of the error curve for the conditions of observation to which he refers his fuzziness #1a.
Semantic	experiment. An LB or YN or MU experiment.
$t_\lambda(u)$	threshold curve for λ, def. 12; a two-valued function of u whose value is 1 inside the quantization interval Δu_λ, and 0 outside this interval. $t_\lambda(u) = P^{excond}(\lambda\|u) = P^{nexcond}(\lambda\|u)$. In contrast, $P^{nexcond}(\lambda\|u^{ex})$ is a fuzzified version of the threshold curve. Loosely speaking, it is identified with the membership curve elicited under exact conditions. See LB,YN-MU assumption of equivalence (Hisdal 1988a).
$u_{\lambda l}, u_{\lambda u}, u_\lambda$	the nonfuzzy lower threshold value in U $u_{\lambda l}$ and upper threshold value $u_{\lambda u}$ of a given subject for classifying an object as being λ in a YN or LB experiment (see def. 11). For extremal concepts (like 'tall', 'small', 'VERY small', see def. 16) only one of these need to be specified. It can then be denoted by u_λ.
u	estimate of the object's attribute value by the subject (see sect. 4).
u^{ex}	exact attribute value of object as measured by the experimenter; e.g. the height in centimeters, measured with a centimeter stick (see sect. 4).
U^{ex}, U	the universe in which u^{ex} and u take on values. In all the formulas and figures we assume a quantized universe (although we often leave out a subscript on the quantized 'u' values in order not to complicate the appearance of the formulas). $u = 165$ cm in a figure should be interpreted as $u \in [160, 170)$ cm. Continuous curves are drawn through the computed points for convenience of visualization (see also end of def. 11).
width	of estimated error curve for a given u^{ex}: size of u interval for which $P^{est}(u\|u^{ex}) > 0$.
YN (yes-no) experiment	an experiment in which a subject answers 'yes' or 'no' to the question of whether an object is λ (def. 3).

Fig. 1b. Notation and Terminology, contnd from fig. 1a

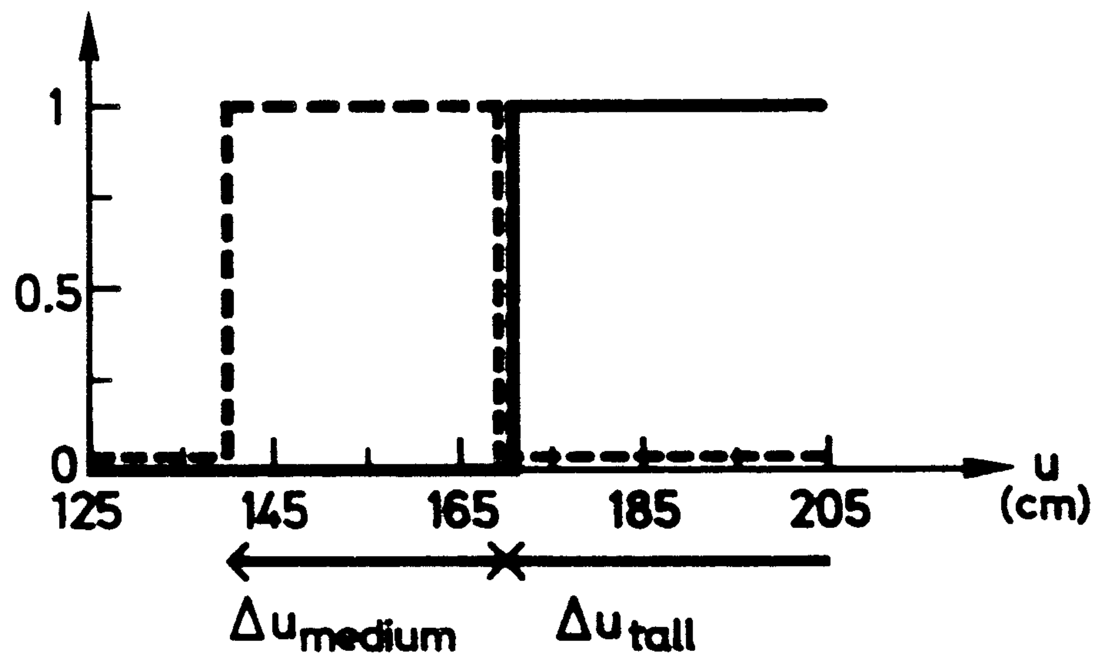

Fig. 2. Possible subjective $t_\lambda(u) = P(\lambda|u)$ threshold or likelihood curves for $\lambda =$tall woman (full curve) and $\lambda =$medium woman (broken curve) respectively. The curves refer to a YN or LB (not a MU) experiment with the reference label set of equation (1) or (16) (only equation (1) for LB-tall). $u =$subject's estimate of the height value of the object. The subject's membership curves, as elicited in a MU experiment, are rounded versions of these curves (see Hisdal 1988a).

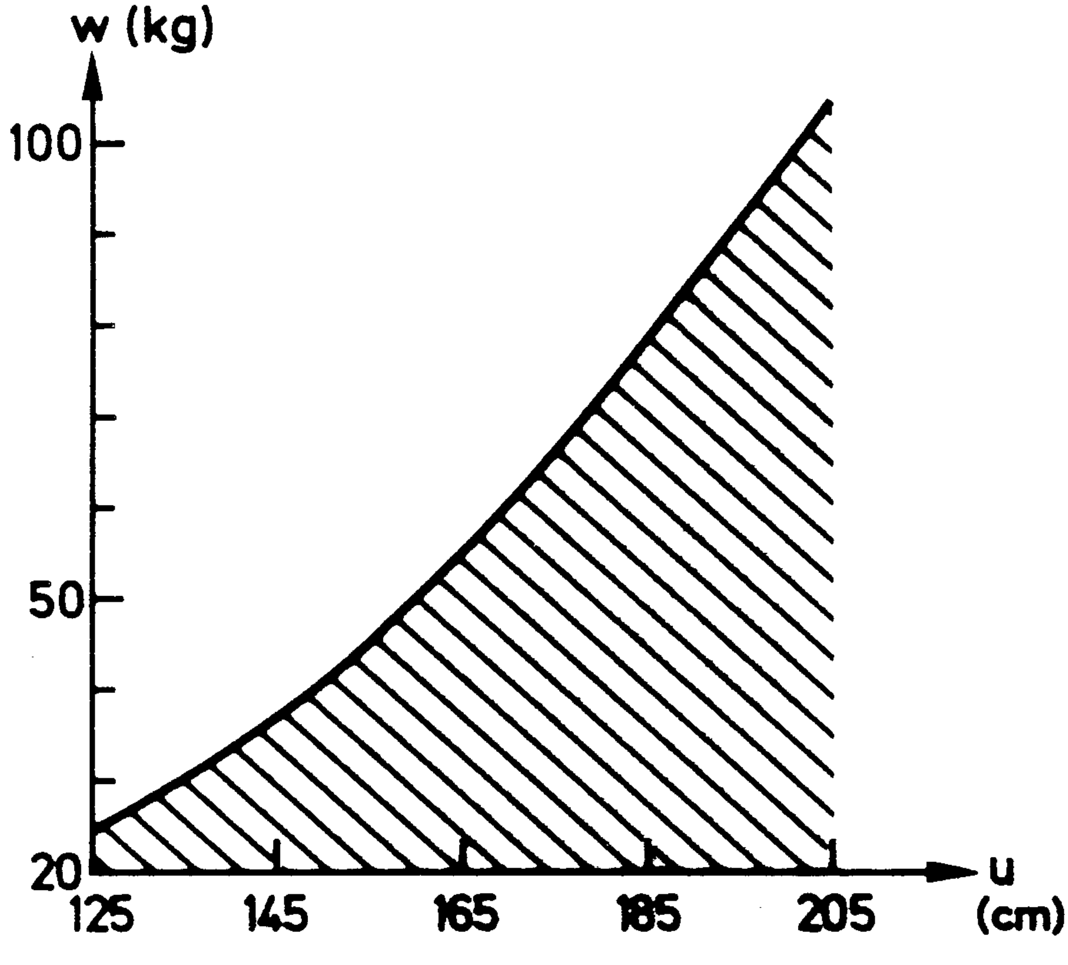

Fig. 3. Shaded area= $\Delta(u,w)_{slim}$ represents a possible subjective quantization interval for 'slim woman' as elicited in a YN experiment (or an LB experiment referring to a label set which does not contain the label 'VERY slim'). u,w=subject's estimate of object's height,weight.

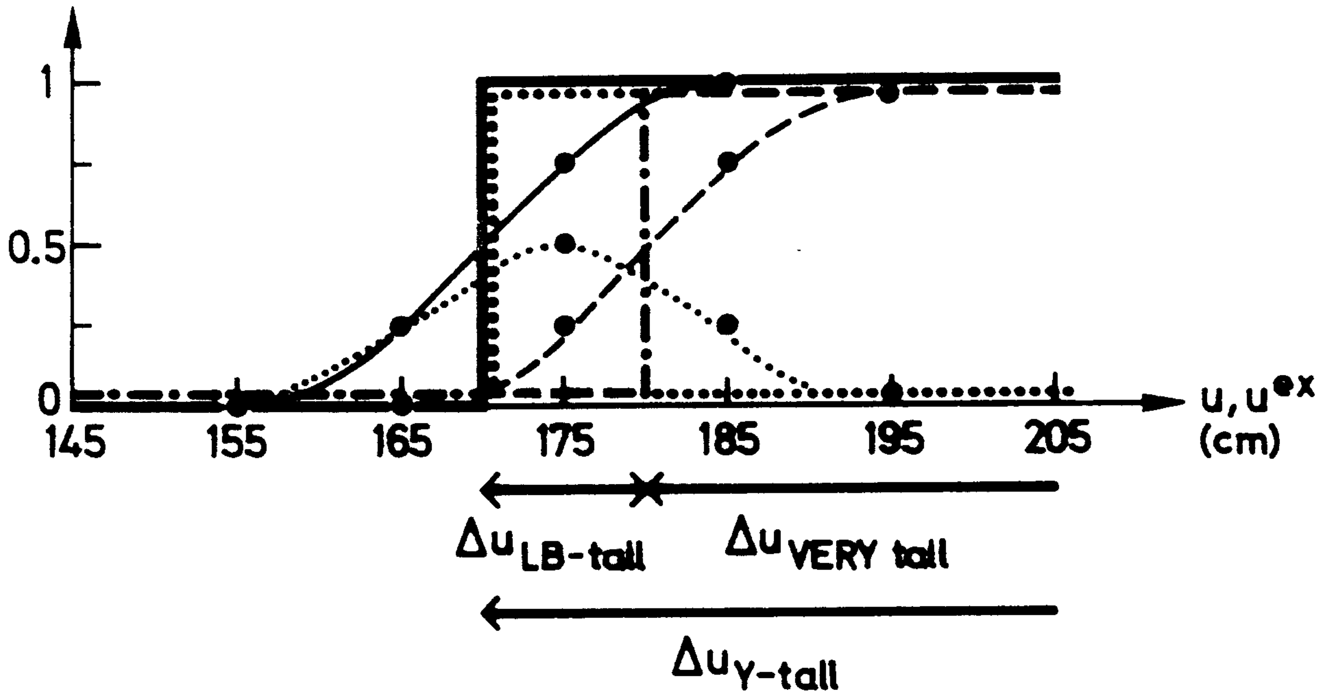

Fig. 4. Threshold and membership curves for 'tall' with YN reference (S-shaped) and with LB reference (bell-shaped) respectively; and for 'VERY tall' with LB or YN reference. Full curves are for λ =tall with YN reference. Broken curves for λ =VERY tall with LB or YN reference. Dotted curves for λ =tall with LB reference and a reference label set which contains the element 'VERY tall'. (When two curve portions run parallel and next to each other, then the full curve represents the correct values for both.) The nonfuzzy $\{0,1\}$ curves represent $P(\lambda|u)$, where u is the subject's estimate of the attribute value of the object. The rounded curves are functions of u^{ex}, the exact attribute value of the object. They represent the $P(\lambda|u^{ex})$ likelihood curves elicited in a nonexact LB or YN experiment. At the same time, they also represent (under the conditions specified in the sequel to def. 17) the $\mu_\lambda(u^{ex})$ membership curves elicited in an exact MU experiment. Note the subnormality of the dotted, rounded LB-tall curve due to an LB-tall quantization interval which is smaller than the width of the $P(u|u^{ex})$ error curve (see sequel to def. 17). Thus for u^{ex} =175 cm, the estimated attribute value u is 185 cm in 25% of all cases, resulting in the assignment to the object of the label 'VERY tall' instead of 'tall'. In another 25% of all cases the estimated attribute value is 165 cm, resulting in the assignment of the label 'medium'. Consequently the label 'tall' will be assigned in only 50% of all cases, even though the value of u^{ex} coincides with the center of the $\Delta u_{LB-tall}$ quantization interval. (Figs. 4-6 refer to quantized universes U, U^{ex}. Continuous curves are drawn through the computed points for purposes of visulization.)

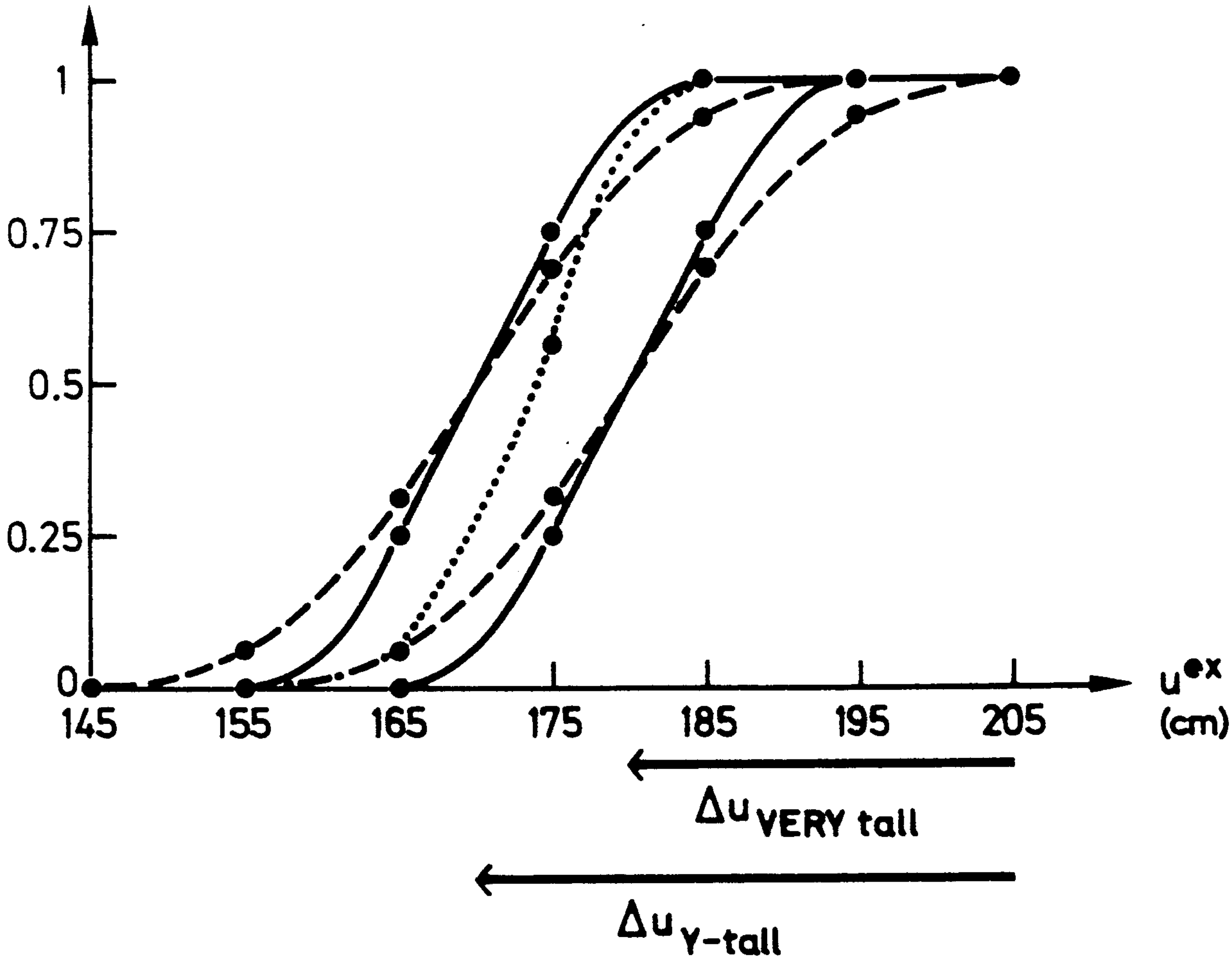

Fig. 5. The 'VERY' operator as a displacement operator according to the TEE model; versus the 'mu-square' operator of traditional fuzzy set theory. Also exact versus nonexact MU experiments. Full curves are the $P^{nexcond}(\lambda|u^{ex}) = \mu_\lambda^{excond}(u^{ex})$ curves of fig. 4 for 'Y-tall' and for 'VERY tall' respectively. (*excond* and *nexcond* signify quantities elicited under exact and nonexact conditions of observation respectively.) Broken curves represent $E^{nexcond}\{\mu_\lambda\}$, the expectation of μ_λ over objects of a given u^{ex}, as elicited in a nonexact MU-experiment for $\lambda =$Y-tall and $\lambda =$ VERY tall respectively (see appendix of Hisdal 1986b, equation (A9)). Note the displacement along the abcissa axis of the curves for 'VERY tall' as compared with those for 'tall'. The dotted curve represents the membership curve of 'VERY tall' as derived from that for 'tall' (full curve), using the 'traditional' fuzzy set mu-square operator for 'VERY'. All the other curves are computed from the TEE model, assuming the subjective quantization intervals marked off in fig. 4. The real and estimated error curves $P(u|u^{ex})$, $P^{est}(u|u^{ex})$ are assumed to be identical (for their numerical values see sequel to def. 17).

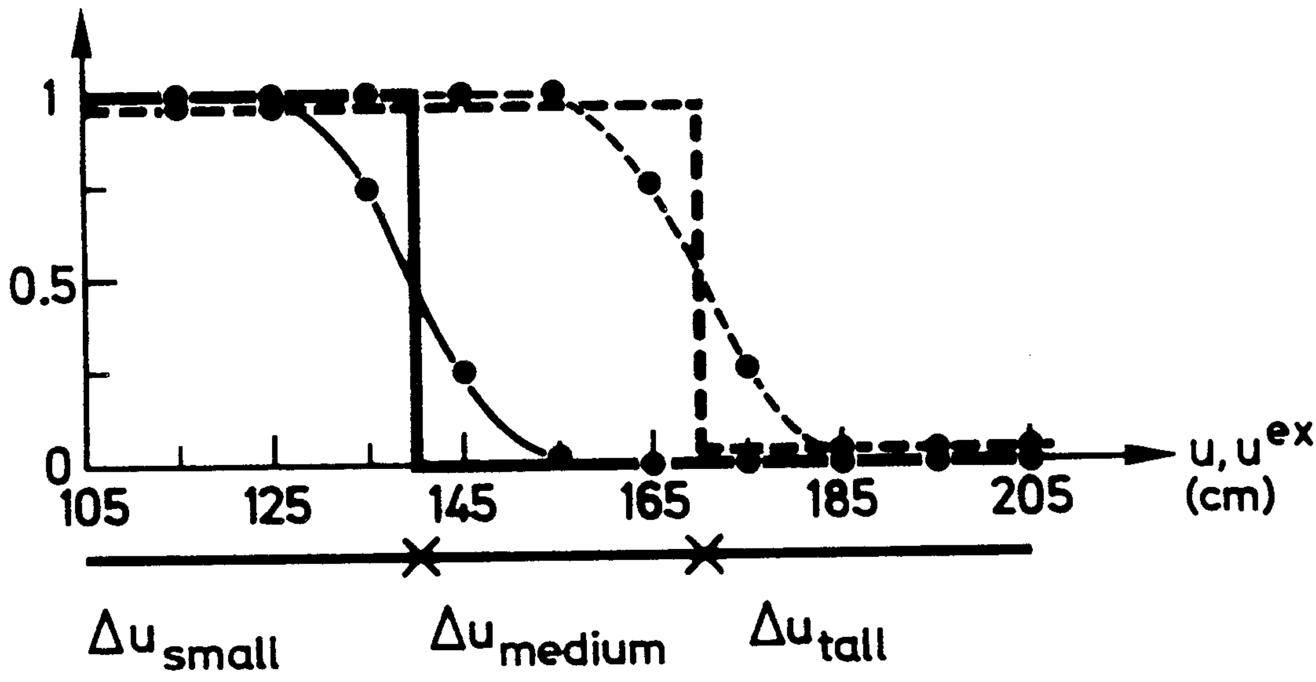

Fig. 6. Negation of 'tall' vs its antonym 'small' according to TEE model. The broken, nonfuzzy, step-shaped curve is the $t(u)$ threshold curve for 'N-tall' ('no' answer to 'tall' in a YN-experiment). Its rounded version is the $\mu(u^{ex})$ memberhip curve for 'N-tall' (membership value concerning a 'no' answer to 'tall' elicited in an exact YN-MU experiment, see sect. 2, def. 5). Full curves: Threshold and membership curves for 'small'. All curves refer to the given quantization intervals and error curve (see caption to fig. 4). Note the displacement of the antonym and negation membership curves relative to each other by an amount equal to Δu_{medium}.

exp1	exp2	exp3	exp4	exp5
LB	YN	YN	YN	LB
Λ	Λ	Λ'	Λ'	Λ'
	$\lambda_{spec} = b$	$\lambda_{spec} = b$	$\lambda_{spec} = a$	
	(=tall)	(=tall)	(=VERY tall)	

Fig. 7. The five semantic experiments to which the definitions of appendix A2 refer. The second row lists the type of experiment (LB or YN), the third row the reference label set (see equations (A8),(A10) and (A12),(A16)), and the last two rows list the specified label of the YN experiments (Is the object λ_{spec}?).