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Report 256,
ISBN 82-73-68-179-3

December 1997

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Abstract

We study the dominant of the convex hull of st-paths with at most k edges in a graph. A complete linear description is obtained for $k \leq 3$ and a class of facet defining inequalities for $k \geq 4$ is given.
Keywords: hop-constrained paths, polyhedra.

1 The k-path polyhedron

In connection with routing in communication networks it may be important to have communication paths with few edges in order to avoid unacceptable delay. A basic problem here is to find a shortest st-path with at most k edges, where k is a specified hop-parameter and where edge weights are nonnegative. This problem, *the k-hop shortest path problem*, may be solved efficiently by dynamic programming using for example a truncated version of the Bellmann-Ford algorithm (see Lawler [4]). The purpose of this note is to make some polyhedral investigations related to this problem.

For constrained shortest path problems (algorithms and applications) we refer to Ahuja et al. [1] and [4]. Exact extended formulations of the problem are studied in Gouveia [3]. In Coullard et al. [2] a closely related problem is studied from a polyhedral point of view (considering directed graphs and walks with exactly k arcs).

Let $G = (V, E)$ be an undirected graph, k a positive integer and s and t two distinct nodes in G. An st-path (i.e., a path in G between s and t with nonrepeating nodes) having *at most* k edges is called a *k-path*, and $\Sigma_k(G)$ is the set of subsets F of E for which the subgraph (V, F) contains a k-path. Consider the *k-path polyhedron*

$$M_k(G) = \text{conv}\{\chi^F : F \in \Sigma_k(G)\} + \mathbb{R}^E_+. \tag{1}$$

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This is the dominant of the convex hull of incidence vectors of k-paths. Throughout we assume that G contains at least one k-path. Moreover, paths are viewed as edge sets.

Recall that an st-cut is an edge set C of the form $C = \delta(W) = \{[i,j] \in E : i \in W,\ j \notin W\}$ where W is a node set containing s but not t. Consider a partition $V_0, \ldots, V_{k+1}$ of V where the sets are nonempty and pairwise disjoint and $s \in V_0$, $t \in V_{k+1}$. Define $T = T(V_0, \ldots, V_{k+1})$ as the set of edges $[u,v]$ where $u \in V_i$ and $v \in V_j$ for some $i < j+1$. We call T a *k-path-cut*. Note that each st-path P in G with $P \cap T = \emptyset$ must contain at least $k+1$ edges, namely one edge in each of the sets $[V_i, V_{i+1}]$ for $i = 0, \ldots, k$.

Lemma 1.1 *Let $F \subseteq E$. Then $F \in \Sigma_k(G)$ if and only if F intersects every cut and every k-path-cut.*

Proof. If $F \in \Sigma_k(G)$, then F clearly intersects every st-cut and, as remarked above, it must also intersect every k-path-cut. To prove the converse, assume that $F \notin \Sigma_k(G)$. If (V, F) does not contain an st-path there must exist an st-cut C with $F \cap C = \emptyset$ and we are done. Otherwise, there is an st-path but all such paths have at least $k+1$ edges. For $i = 0, \ldots, k$ let V_i consist of the nodes with distance i from s ("distance" means minimum number of edges in a path joining s and the node in question). Let $V_{k+1} = V \setminus \cup_{i=0}^{k} V_i$ and observe that $t \in V_{k+1}$. By construction there is no edge in F joining a node in V_i and a node in V_j where $j > i+1$, i.e., $F \cap T(V_0, \ldots, V_{k+1}) = \emptyset$ and the proof is complete. □

A consequence is that a valid integer linear programming formulation of the shortest k-path problem with nonnegative weights c_{ij} for $[i,j] \in E$ is: minimize $\sum_{[i,j]\in E} c_{ij} x_{ij}$ subject to (i) $\mathbf{x}(C) := \sum_{[i,j]\in C} x_{ij} \geq 1$ for each st-cut C, (ii) $\mathbf{x}(T) \geq 1$ for each k-path-cut T, and (iii) $x_{ij} \in \{0,1\}$ (or $x_{ij} \geq 0$ and integer). We call each inequality in (i) resp. (ii) a *cut inequality* resp. a *k-path-cut inequality*.

Example. Consider the complete graph on 5 nodes $v_0, \ldots, v_4$ with $s = v_0$, $t = v_4$ and $k = 3$. Then the valid inequality $\mathbf{x}(T) \geq 1$ where $T = E \setminus \{[v_0,v_1], [v_1,v_2], [v_2,v_3], [v_3,v_4]\}$ is a 3-path-cut inequality corresponding to the choice $V_i = \{v_i\}$ for $i = 0, \ldots, 4$. In fact, a complete linear description of the 3-path polyhedron for this graph consists of nonnegativity constraints, cut inequalities and 3-path-cut inequalities.

2 Completeness for $k \leq 3$

Up to scaling there is a unique minimal linear system of inequalities with solution set $M_k(G)$. This follows from the fulldimensionality of the polyhedron (recall that G is assumed to have a k-path). Each inequality in this system which is not a nonnegativity constraint has the form $\sum_{e\in E} a_e x_e \geq \alpha$ where α and a_e for each $e \in E$ are integral and $a_e \geq 0$ and $\alpha \geq 1$. These properties hold for all G and k. We next determine a complete linear description of $M_k(G)$ for arbitrary

G but with $k \leq 3$. For $k = 1$ one easily proves that $M_1(G)$ is the solution set of $x_{st} \geq 1$, $x_e \geq 0$ for each $e \in E \setminus \{st\}$.

Define a 2-*star* as a subset of E of the form $T(S_1, S_2) = \{[s,t]\} \cup \{[s,v] : v \in S_1\} \cup \{[t,v] : v \in S_2\}$ where $S_1 \cup S_2 = V \setminus \{s,t\}$ and $S_1 \cap S_2 = \emptyset$. In particular, the stars $\delta(s)$ (all edges incident to s) and $\delta(t)$ are 2-stars. Moreover $T \subseteq E$ is a 2-star if and only if T is either a star or a 2-path-cut $T(V_0, \ldots, V_3)$ with $V_0 = \{s\}$ and $V_3 = \{t\}$. It follows that for each 2-star T the 2-*star inequality* $\mathbf{x}(T) \geq 1$ is valid for $M_2(G)$.

Proposition 2.1 *For any graph G a complete linear description of $M_2(G)$ consists of the nonnegativity constraints and the 2-star inequalities $\mathbf{x}(T) \geq 1$ for each 2-star T.*

Proof. Each 2-path is either the single edge $[s,t]$ or of the form $[s,v],[v,t]$ for some $v \in V \setminus \{s,t\}$. Let E_1 be the union of these edge sets, and define the corresponding subgraph $G_1 = (V, E_1)$. It is easy to see that a complete linear system for $M_2(G)$ consists of the inequalities $x_e \geq 0$ for each $e \in E \setminus E_1$ together with the inequalities in a complete linear description of $M_2(G_1)$. Note that in G_1 *all st*-paths are 2-paths and it is well known that a complete linear description of the dominant of the convex hull of *st*-paths (in any graph) consists of nonnegativity constraints and cut constraints. But we observe that cut constraints in G_1 coincide with 2-star inequalities in G, and the proof is complete. □

This result may also be obtained after some calculation using projection techniques (as Fourier-Motzkin elimination). If U and W are node sets we denote the set of edges with one end node in U and the other in W by $[U,W]$. A point $\bar{x}$ is called a *root* of an inequality $\mathbf{a}^T\mathbf{x} \geq \alpha$ if $\mathbf{a}^T\bar{x} = \alpha$.

Proposition 2.2 *For any graph G a complete linear description of $M_3(G)$ consists of the nonnegativity constraints, the cut inequalities and the 3-path-cut inequalities.*

Proof. For notational simplicity we assume that G is a complete graph. Let $\mathbf{a}^T\mathbf{x} \geq \alpha$ be a facet defining inequality for $M_3(G)$ which is not a nonnegativity constraint. As remarked above, we may assume that $a_e \geq 0$ and $\alpha \geq 1$ are integral. Let M^* be the induced facet. Define $V_1 = \{v \in V : a_{sv} = 0\}$, $V_2 = \{v \in V : a_{tv} = 0\}$ and $V_3 = \{v \in V : a_{sv} > 0,\ a_{tv} > 0\}$. Then these three sets are a partition of $V \setminus \{s,t\}$ (for if $v \in V_1 \cap V_2$ then $\alpha = 0$). Moreover, $a_{st} = \alpha$ as validity of $\mathbf{a}^T\mathbf{x} \geq \alpha$ implies $a_{st} \geq \alpha$ and if this inequality were strict each point in M^* would satisfy $x_{st} = 0$ (contradicting that $M_3(G)$ is fulldimensional and M^* a facet). Similarly, we obtain $a_e = \alpha$ for each $e \in [V_1, t] \cup [s, V_2] \cup [V_1, V_2]$.

Let $W = V_1 \cup \{s\}$. If there is no edge $e \in [V_1, V_3]$ with $a_e = 0$ we see that each root of $\mathbf{a}^T\mathbf{x} \geq \alpha$ satisfies $\mathbf{x}(\delta(W)) = 1$ and since M^* is a facet we conclude that $\mathbf{a}^T\mathbf{x} \geq \alpha$ is a positive multiple of $\mathbf{x}(\delta(W)) \geq 1$. Alternatively, there exists an $e \in [V_1, V_3]$, say $e = [u,v]$ with $u \in V_1$, $v \in V_3$ and $a_e = 0$.

We claim that $a_{sv} = a_{vt} = \alpha$ and $a_{vw} = 0$ for all $w \in V_1 \cup V_2$. From the 3-path $[s,u],[u,v],[v,t]$ we see that $a_{vt} \geq \alpha$ and equality holds for the same reason as given in the first paragraph of the proof. Let $w \in V_1 \setminus \{u\}$. The edge $[v,w]$ must lie in some root and since $a_{sv} > 0$ and $a_{wt} = \alpha$, the only possible choice is the 3-path $[s,w],[w,v],[v,t]$. From this we conclude that $a_{vw} = 0$ for all $w \in V_1$. Let $W = V_1 \cup \{s,v\}$. If each root of $\mathbf{a}^T\mathbf{x} \geq \alpha$ satisfies $\mathbf{x}(\delta(W)) = 1$ we are done (as above), so we may assume that there is an $F \in \Sigma_3(G)$ with more than one edge in $\delta(W)$ and with $\sum_{e \in F} a_e = \alpha$. It is easy to see that the only possibility is that F contains a 3-path $[s,w],[w,v],[v,t]$ for some $w \in V_2$. This implies that $a_{vw} = 0$. From this we obtain, as above, that $a_{sv} = \alpha$ and also that $a_{vv'} = 0$ for all $v' \in V_2$. This proves the claim.

Finally we prove that $V_3 = \{v\}$ (where v was defined above). Assume not, and let $w \in V_3 \setminus \{v\}$. Consider the edge $e = [v,w]$. Since $a_{sv} = a_{sw} = a_{vt} = a_{wt} = \alpha$ there is no root containing e; a contradiction. Thus $V_3 = \{v\}$ and the inequality $\mathbf{a}^T\mathbf{x} \geq \alpha$ is a positive multiple of a 3-path-cut inequality. □

Thus, for $k \leq 3$ and for all graphs G nonnegativity constraints, cut inequalities and k-path-cut inequalities are sufficient to describe $M_k(G)$, and we note that all these inequalities are rank inequalities. This is not true for $k \geq 4$ and general graphs as seen next.

Let $n \geq 3$ and define G_n to be the graph with nodes $v_0, \ldots, v_n, w_1, \ldots, w_n$ and edges $[v_{i-1}, w_i]$, $[w_i, v_i]$ and $[v_{i-1}, v_i]$ for $i = 1, \ldots, n$. Consider the *triangle-path* inequality

$$\sum_{i=1}^{n} x_{v_i v_{i+1}} \geq n - 1. \tag{2}$$

This inequality is valid for $M_{n+1}(G_n)$ (so here $k = n+1$) and it is easy to verify that it defines a facet of $M_{n+1}(G_n)$. Let G be a graph obtained from G_n by adding some edges. Then (2) may be lifted (using standard techniques, see Nemhauser and Wolsey [5]) to obtain a facet for $M_{n+1}(G)$. Such a facet has right hand side $n-1$ and coefficients lying in the set $\{0, 1, \ldots, n-1\}$. As an example, if $n = 4$ and edges are added so G is a complete graph, then such a lifted inequality has coefficients 0, 1, 2 and 3. Thus, for $k \geq 4$, the facial structure of the polyhedron $M_k(G)$ may be rather complex.

Acknowledgment. The author wishes to thank Luis Gouveia for interesting discussions in connection with this work.

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