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STABLE SET POLYTOPES FOR A CLASS OF CIRCULANT GRAPHS

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Abstract. We study the stable set polytope $P(G_n)$ for the graph G_n with n nodes and edges $[i, j]$ when $|i-j| \leq 2$ (using modulo n calculation); this graph coincides with the anti-web $\bar{W}(n, 3)$. A minimal linear system defining $P(G_n)$ is determined. The system consists of certain rank inequalities with some number theoretic flavour. A characterization of the vertices of a natural relaxation of $P(G_n)$ is also given.

Keywords: Polyhedral combinatorics, stable sets, circulant graphs.

1. Introduction. Let $n \geq 3$ be a positive integer and let $\mathbf{C}_n = (c_{i,j}) \in \mathbb{R}^{n,n}$ be the $(3, n)$-circulant matrix, i.e, for $i = 1, \ldots, n$ we have $c_{i,j} = 1$ if $|i - j| \leq 2$ and $c_{i,j} = 0$ otherwise. We use modulo n calculation for indices, so e.g. $n + 1$ and 1 are identified. We let **0**, **1** and **2** denote a vector of suitable dimension with all components being 0,1 and 2, respectively. In this paper we are concerned with the polytope

$$P_n = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{C}_n\mathbf{x} \leq \mathbf{1}, \ \ \mathbf{x} \geq \mathbf{0}\} \tag{1.1}$$

and its integer hull, i.e. the convex hull of the integral points in P_n. These objects relate to stable sets. Let $V = \{1, \ldots, n\}$ and consider the *circulant graph* $G_n = (V, E)$ with node set V and edge set E consisting the edges $[i, i + 1]$ and $[i, i + 2]$ for $i \in V$. It is useful to imagine the nodes of V placed consecutively along a circle so node 1 and n are adjacent. The graph G_n coincides with the anti-web $\bar{W}(n, 3)$. We recall from [10] that the *web* $W(n, k)$ is defined as the graph on n nodes with edges $[i, j]$ for $j = i + k, \ldots, i + n - k$. The *anti-web* $\bar{W}(n, k)$ is the complement of $W(n, k)$, i.e., the graph on n nodes containing precisely those edges that are not in $W(n, k)$. A stable set in a graph is a subset S of nodes such that no pair of nodes in S are adjacent. A stable set S in G_n is a set of nodes such that the distance between consecutive nodes is at least 3. Let $\mathcal{S}_n$ denote the set of all stable sets in G. Then the integral points in P_n coincides with the incidence vectors of sets in $\mathcal{S}$, so the integer hull of P_n equals the *stable set polytope* $P(G_n)$ associated with the graph G_n. P_n may be viewed as the relaxation of $P(G_n)$ consisting of nonnegativity constraints and clique

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constraints. Note that $P(G_n)$ is full-dimensional as it contains the origin and the coordinate vectors. Moreover, each nonnegativity constraint $x_j \geq 0$ defines a facet of $P(G_n)$ which is called a *trivial facet.* Each clique inequality $x_i + x_{i+1} + x_{i+2} \leq 1$ also defines a facet of $P(G_n)$ and we call it a *clique facet.*

The purpose of this paper is to study the polytopes P_n and $P(G_n)$. We determine all the vertices of P_n and a minimal linear system of inequalities defining $P(G_n)$. This system contains, apart from the inequalities defining P_n, certain inequalities with $(0,1)$-coefficients called the 1-*interval inequalities.* These inequalities are of interest for stable sets in general graphs as well as they produce facets via the procedure of lifting. The work was motivated by a study of integer programming models for finding minimum cost spanning trees satisfying a "2-hop constraint", see [2]. This constraint says that each node or one of its neighbors is adjacent to a given root node. The results of the present paper are useful for finding strong relaxations for 2-hop spanning tree problems.

Consider a weighted stable set problem in G_n: for given numbers $w_j, j \in V$ find a stable set S in G_n with $\sum_{j \in S} w_j$ smallest possible. This problem may be solved in polynomial time as follows. Choose $j \in V$. Any stable set S satisfies either (i) $j \in S$, (ii) $j+1 \in S$, or (iii) $j, j+1 \notin S$. Thus the weighted stable set problem may be solved by finding an optimal stable set for each of these three cases and comparing the solutions. Each of the three subproblems may be solved by linear programming since deleting proper columns results in a totally unimodular coefficient matrix. The existence of an efficient algorithm for solving the weighted stable set problem in G_n is a motivation for looking for a complete linear description of $P(G_n)$.

A survey of the stable set problem and stable set polytopes is given in [4] (Chapter 9). Complete linear description of stable set polytopes are known for certain classes of graphs, like bipartite graphs, interval graphs and chordal graphs; all these classes are perfect graphs so nonnegativity constraints and clique constraints suffice to describe the corresponding stable set polytopes. Furthermore, for series-parallel graphs the stable set polytopes are described by nonnegativity constraints, edge constraints and odd circuit constraints, for a proof see [7]. For graph theory and polyhedral theory used in the paper, see [9] and [8]. A $(0,1)$-matrix is called an *interval matrix* provided that in each row the ones occur consecutively. A well-known fact is that every interval matrix is totally unimodular (see [8]). If $\mathbf{a}^T\mathbf{x} \leq \alpha$ is valid inequality for a polytope

P we say that each point in $P \cap \{\mathbf{x} : \mathbf{a}^T\mathbf{x} = \alpha\}$ is a *root* of the inequality $\mathbf{a}^T\mathbf{x} \leq \alpha$ (or the corresponding face of P).

2. The polytope P_n. It is clear that the incidence vector of each stable set in G_n is a vertex of P_n. In this section we determine the remaining vertices.

Certain subsets of the node set V are of interest in the following. We shall call a subset of consecutive nodes in V an *interval* and note that e.g. $\{n-1, n, 1\}$ is an interval (the modulo n calculation). Any *strict* subset T of V corresponds to a partition of V into nonempty, consecutive, disjoint intervals $I_1, J_1, I_2, J_2, \ldots, I_t, J_t$ where $T = \cup_{s=1}^t I_s$. Note that the intervals J_s are determined by the intervals I_s. We then write $T = I_1 + \ldots + I_t$. A 1-*interval set* is a subset T being the union of intervals $I_1, \ldots, I_t$ separated by just one node, i.e., $|J_s| = 1$ for all $s \leq t$.

Consider a 1-interval set $T = I_1 + \ldots + I_t$ satisfying $|I_s| \in \{1, 2\}$ for $s \leq t$. Associated with T is the point $\mathbf{x}^T \in \mathbb{R}^n$ given by $x_j^T = 1/2$ for $j \in T$ and $x_j^T = 0$ otherwise, i.e., $\mathbf{x}^T = (1/2)\chi^T$. A point $\mathbf{x}^T$ for which t (also equal to the number of zeros in $\mathbf{x}^T$) is odd will be called an *odd* 1/2-*string*.

PROPOSITION 2.1. *The vertices of P_m are the incidence vectors of stable sets in G_n, all odd 1/2-strings, and, provided that n is not a multiple of 3, the vector with all components equal to* 1/3.

Proof. Let $\mathbf{x}$ be a *nonintegral* vertex of $P(G_n)$. We establish several properties of $\mathbf{x}$, eventually showing that $\mathbf{x}$ must be an odd 1/2-string or have all components equal to 1/3.

Property 1: For each $i \leq n$ we have that $x_i < 1$ and that either x_i or x_{i+1} is positive. If $x_i = x_{i+1} = 0$, the $(m-2)$-dimensional vector $\mathbf{x}'$ with the remaining components of $\mathbf{x}$ must be a vertex of the polytope defined by $\mathbf{C}'\mathbf{x}' \leq \mathbf{1},\ \mathbf{x}' \geq \mathbf{0}$ where $\mathbf{C}'$ is the matrix obtained from $\mathbf{C}$ by deleting columns i and $i+1$. But $\mathbf{C}'$ is an interval matrix and therefore totally unimodular. This implies that $\mathbf{x}'$ is integral (in fact $(0,1)$) and so is $\mathbf{x}$; a contradiction. Therefore, either x_i or x_{i+1} is positive. Similarly, if $x_i = 1$, then $x_{i-2} = x_{i-1} = x_{i+1} = x_{i+2} = 0$, and the remaining components are determined from an interval matrix (deleting the columns $i-2, \ldots, i+2$ and one row in $\mathbf{C}$) and again we arrive at the desired contradiction.

By Property 1 the support T of $\mathbf{x}$ (i.e. the indices of nonzero components) is either V or a 1-interval set $T = I_1 + \ldots + I_t$. Consider first the case when $T = V$. Then all variables in $\mathbf{x}$ are positive and therefore $\mathbf{C}_n\mathbf{x} = \mathbf{1}$ (as there must be n active inequalities). But $\mathbf{C}_n$ is nonsingular if and only if n is not a multiple of 3, and in that case we see that $\mathbf{x} = (1/3, \ldots, 1/3)$. In the remaining part of the proof we may assume that $T \neq V$.

Property 2: If $i \notin T$ then the equation $x_{i-1} + x_i + x_{i+1} = 1$ holds. Otherwise we would again have that all components in $\mathbf{x}$ except the ith were determined by an interval matrix (deleting the assumed nonactive constraint), and a contradiction arises.

Property 3: $|I_s| \leq 2$ for all $s \leq t$. To prove this we determine an upper bound on the number of nonredundant, active clique inequalities in $\mathbf{x}$. If $|I_s| = 1$, say $I_s = \{i\}$, then the clique inequality $x_{i-1} + x_i + x_{i+1} \leq 1$ is not active because that would give $x_i = 1$ (as $x_{i-1} = x_{i+1} = 0$); a contradiction due to Property 1. If $|I_s| = 2$, say $I_s = \{i, i+1\}$, then $x_{i-1} + x_i + x_{i+1} \leq 1$ and $x_i + x_{i+1} + x_{i+2} \leq 1$ are equivalent so one of them is redundant. Finally, if $|I_s| > 2$, say $I_s = \{i, i+1, \ldots, j\}$, then none of the inequalities $x_{i-1} + x_i + x_{i+1} \leq 1$ and $x_{j-1} + x_j + x_{j+1} \leq 1$ are active for that would give that either x_{i+2} or x_{j-2} were 0. Note that all the mentioned inactive or redundant inequalities are distinct. Let m_1 and m_2 be the number of intervals I_s with $|I_s|$ equal to 1 and 2, respectively. The number m_3 of intervals I_s with $|I_s| \geq 3$ clearly satisfies $m_3 = t - m_1 - m_2$. Our discussion shows that an upper bound on the number of active, nonredundant clique inequalities in $\mathbf{x}$ is $n - (m_1 + m_2 + 2m_3) = n - t - m_3$. But $\mathbf{x}$ has $n - t$ positive components to be determined by the active clique constraints, so

$$n - t - m_3 \geq n - t$$

which implies that $m_3 = 0$ and Property 3 follows.

The counting argument just given also shows that, except for those special constraints mentioned in the paragraph above, *all other* clique constraints are active in $\mathbf{x}$. From this we deduce that the nonzero components of $\mathbf{x}$ must alternate between α and $1 - \alpha$ where $0 < \alpha < 1$. The number of nonzeros $n - t$ must be odd, otherwise we could write $\mathbf{x}$ as the midpoint of two solutions in P_n similar to $\mathbf{x}$ (with α replaced by $\alpha - \epsilon$ and $\alpha + \epsilon$, respectively for suitably small ϵ). Finally, as $n - t$ is odd, one of the active clique inequalities gives that $\alpha = 1 - \alpha$, i.e., $\alpha = 1/2$. This means that $\mathbf{x}$ is an odd 1/2-string and the proof is complete.

□

3. Rank facets of $P(G_n)$. In this section we study the stable set polytope $P(G_n)$ and valid inequalities for $P(G_n)$ of the form $x(T) \leq \alpha$ for $T \subseteq V$; such inequalities are called *rank (or canonical) inequalities*. Clearly, we may restrict the attention to $\alpha = \alpha(T) := \max\{|S \cap T| : S \text{ is a stable set in } G_n\}$ which is the stability number in the subgraph of G_n induced by T.

First we consider how to compute $\alpha(T)$ for a given subset T of V. In [6] a polynomial algorithm is given for computing the stability number of a *claw-free graph*, i.e., a graph with no induced subgraph isomorphic to the star $K_{1,3}$. The algorithm is based on a reduction to a matching problem. Since G_n is claw-free the subgraph $G_n[T]$ is also claw-free and the algorithm of [6] could be used to determine $\alpha(T)$. However, the special structure of G_n makes it possible to determine $\alpha(T)$ by a simple greedy algorithm which is discussed in the following.

Let $\mathbf{A} \in \mathbb{R}^{m,n}$ be a $(0,1)$-matrix. Following [3] we say that $\mathbf{A}$ is *greedy* if the greedy algorithm correctly solves the linear program

$$\max\{\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{0} \leq \mathbf{x} \leq \mathbf{u}\} \tag{3.1}$$

for all $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c}, \mathbf{u} \in \mathbb{R}^n$ with $c_1 \geq c_2 \geq \ldots \geq c_n$. The greedy algorithm for (3.1) determines a solution $\mathbf{x}'$ as follows: for $j = 1, \ldots, n$ let x'_j be the maximum real number r such that $(x'_1, \ldots, x'_{j-1}, r, 0, \ldots, 0)$ is feasible in (3.1). Note that $\mathbf{x}'$ is integral whenever both $\mathbf{u}$ and $\mathbf{b}$ are integral, so the greedy algorithm also solves the integer LP corresponding to (3.1). It was shown in [5] that $\mathbf{A}$ is greedy if and only if neither of the following two submatrices is a submatrix of $\mathbf{A}$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

An immediate consequence of this result is that every interval matrix is greedy. Now, consider the circulant matrix $\mathbf{C}_n$ defined in the introduction. The matrix $\mathbf{C}'$ obtained from $\mathbf{C}_n$ by deleting columns j and $j+1$ and permuting the columns suitably is an interval matrix and therefore greedy. In particular, for each $\mathbf{u} \in \{0,1\}^{n-2}$ we can solve the integer program

$$\max\{\mathbf{1}^T\mathbf{x}' : \mathbf{C}'\mathbf{x}' \leq \mathbf{1},\ \mathbf{0} \leq \mathbf{x}' \leq \mathbf{u},\ \mathbf{x}' \text{ is integral}\} \tag{3.2}$$

by the greedy algorithm. We see that (3.2) is the stable set problem in the subgraph of G_n induced by the nodes $\{k : u_k = 1 \text{ and } k \neq j, j+1\}$.

For $T \subseteq V$ and $j \in T$ define

$$\alpha_j(T) = \max\{|S \cap T| : S \in \mathcal{S}_n,\ j \in S\}. \tag{3.3}$$

The following greedy algorithm determines $\alpha_j(T)$: initially let $S = \{j\}$, $s := j$ and choose $k \in \{s+3, \dots, j-3\}$ "smallest possible" with $k \in T$ and add k to S. Repeat this process for $s := k$ until no more k can be found. The correctness of this algorithm follows from the discussion above.

We can calculate $\alpha(T)$ as follows. If $T = V$, we obtain $\alpha(T) = \alpha(G_n) = \lfloor n/3 \rfloor$. Assume next that $T \neq V$, say $j \in T$ but $j+1 \notin T$. We find a solution of $\alpha' = \max\{|S \cap T| : S \in \mathcal{S}_n,\ j \notin S\}$ by removing nodes j and $j+1$ from the graph and using the greedy algorithm in the interval graph we then obtain with start in $j+2$. We then calculate $\alpha_j(T)$ using the greedy algorithm above and conclude that $\alpha(T) = \max\{\alpha', \alpha_j(T)\}$. We call this procedure for finding $\alpha(T)$ the α-GREEDY algorithm (with start in node j). It is used in some proofs later.

We now calculate $\alpha(T)$ for certain interesting 1-interval sets.

LEMMA 3.1. *Let $T = I_1 + \ldots + I_t$ be a 1-interval set satisfying, for $s = 1, \dots, t$, $|I_s| \equiv 1 (\bmod 3)$, say $|I_s| = 3k_s + 1$ where k_s is a nonnegative integer. Then $\alpha(T) = \sum_{s=1}^t k_s - \lfloor t/2 \rfloor$, or equivalently, $\alpha(T) = n/3 - t/6$ when t is even and $\alpha(T) = n/3 - t/6 - 1/2$ when t is odd.*

Proof. The result may be found by the α-GREEDY algorithm, but we give an alternative proof here. Let $s \in \{1, \dots, t\}$. We note that $\alpha(I_s) = k_s + 1$ and that there is a unique maximum stable set S in I_s and, moreover, S contains both the end points of the interval I_s. The next observation is that $\alpha(I_s \cup I_{s+1}) = k_s + k_{s+1} + 1$ and every stable set in $I_s \cup I_{s+1}$ must contain both the end points of (exactly) one of the two intervals. One such stable set, say S_s, contains k_s nodes in I_s (and therefore the two end points) but it does not contain the "right-hand end node" of I_{s+1}. Assume now that t is even and let S be the union of such sets $S_1, S_3, \dots, S_{t-1}$. From the construction we see that S is a stable set in G_n. Furthermore, $|S| = \sum_{s=1}^t k_s + t/2$ so we conclude that $\alpha(T) \geq \sum_{s=1}^t k_s + t/2$. Moreover, this is an equality for otherwise, for some s, I_s and I_{s+1} would contain $k_s + 1$ and $k_{s+1} + 1$ nodes respectively; a contradiction as explained above. Using that $\sum_{s=1}^t (3k_s + 2) = n$ we conclude that $\alpha(T) = n/3 - t/6$ for t even. Finally, if t is odd, similar arguments lead to $\alpha(T) = \sum_{s=1}^t k_s + (t-1)/2 = n/3 - t/6 - 1/2$ and the proof is complete. □

Recall that when $T = V$ we have $\alpha(T) = \alpha(G_n) = \lfloor n/3 \rfloor$. Therefore the inequality

$$\mathbf{x}(V) \leq \lfloor n/3 \rfloor \tag{3.4}$$

is valid for $P(G_n)$; this is the anti-web inequality introduced in [10]. It is easy to see that this inequality is nonredundant if and only if n is not a multiple of 3. In the remaining discussion we consider rank inequalities $x(T) \leq \alpha(T)$ for which $T \neq V$.

LEMMA 3.2. *Let $T = I_1 + \ldots + I_t$ be a strict subset of V (where I_s are disjoint intervals) such that $x(T) \leq \alpha(T)$ is a facet of $P(G_n)$ different from each trivial and clique facet. Then the following holds*

$$\begin{array}{ll} \text{(i)} & T \textit{ is a } 1\textit{-interval set;} \\ \text{(ii)} & |I_s| \equiv 1 \pmod 3 \;\; \textit{for } s = 1, \ldots, t; \\ \text{(iii)} & t \textit{ is odd and } t \geq 3. \end{array} \tag{3.5}$$

Proof. (i) Let F be the facet of $P(G_n)$ defined by $x(T) \leq \alpha(T)$. Assume that $i, i+1 \notin T$ for some $i \leq n$. We may assume that $i - 1 \in T$ (otherwise another i could be chosen). Consider the clique $K = \{i-3, i-2, i-1\}$. Since F is not a clique facet, F has a root S with $S \cap K = \emptyset$. Note that $S \setminus \{i, i+1\}$ is also a root of F as $i, i+1 \notin T$, so we may assume that $i, i+1 \notin S$. Let $S' = S \cup \{i-1\}$ and observe that S' is a stable set in G_n. But $\chi^{S'}(T) = \chi^S(T) + 1 = \alpha + 1$ which contradicts the validity of $x(T) \leq \alpha(T)$. Thus, for each $i \leq n$, T contains either i or $i+1$, and therefore T is a 1-interval set.

(ii) Assume that $|I_s| \equiv 2 \pmod 3$ and let $I_s = \{l, \ldots, r\}$. Using the α-GREEDY algorithm with start in node $l-2$ it is easy to see that $\alpha(T) = \alpha(T \cup \{r+1\})$ (as we find an optimal stable set not containing the node $r+1$). But then $x(T) \leq \alpha(T)$ is the sum of the two valid inequalities $x(T \cup \{r+1\}) \leq \alpha(T \cup \{r+1\})$ and $-x_{r+1} \leq 0$ which contradicts that $x(T) \leq \alpha(T)$ defines a facet of $P(G_n)$. This proves that $|I_s| \not\equiv 2 \pmod 3$.

Assume next that $|I_s| \equiv 0 \pmod 3$, say $|I_s| = 3k$ for some $k \geq 1$. Let $I_s = \{l, \ldots, r\}$. There is a root S of $x(T) \leq \alpha(T)$ such that $S \cap I_s$ consists of the k nodes $l+1, l+4, \ldots, r-1$. Note that $S \cap \{l-1, l, r, r+1\} = \emptyset$. Thus the incidence vector of $S \setminus I_s$ must maximize $\mathbf{x}(T \setminus I_s)$ over the set of stable sets in G_n. Therefore $\mathbf{x}(T \setminus I_s) \leq |(S \setminus I_s) \cap T|$ is a valid inequality for $P(G_n)$. But $\mathbf{x}(I_s) \leq k$ is clearly a valid inequality as well, and if we add these two inequalities we obtain $x(T) \leq |S| = \alpha(T)$. This contradicts that $x(T) \leq \alpha(T)$ is nonredundant, and Property (ii) follows.

(iii) Assume that t is even. From the proof of Lemma 3.1 (and Property (ii)) it is clear that $\alpha(T)$ equals the sum of the stability numbers $\alpha(I_s \cup I_{s+1})$ for all $s \leq t$ being odd. As above this means that the rank inequality $x(T) \leq \alpha(T)$ is redundant; a contradiction. Therefore, t must be odd. Furthermore, one can check that $\alpha(V \setminus \{i\}) = \alpha(V)$ for each $i \in V$. This implies that the rank inequality $\mathbf{x}(V \setminus \{i\}) \leq \alpha(V \setminus \{i\})$ is redundant as it is the sum of the rank inequality $\mathbf{x}(V) \leq \alpha(V)$ and the inequality $-x_i \leq 0$. Therefore t can not be 1, so $t \geq 3$. □

Our next result characterizes *all* the nonredundant rank inequalities in an explicit way.

THEOREM 3.3. *Let $T = I_1 + \ldots + I_t \subseteq V$ be a strict subset of V. Then the rank inequality $x(T) \leq \alpha(T)$ defines a facet of $P(G_n)$ if and only if (3.5) holds.*

Proof. Due to Lemma 3.2 we only need to show the sufficiency of the conditions. So, assume that (3.5) holds. Let $|I_s| = 3k_s + 1$ for $s \leq t$. If S is a stable set in G_n and $|S \cap I_s| = k_s + 1$ ($|S \cap I_s| = k_s$) we say that I_s is *closed* (*open*). From the proof of Lemma 3.1 and (3.5) it follows that a stable set is a root of $x(T) \leq \alpha(T)$ if and only if the intervals I_s alternate between being closed and open except for one s where both I_s and I_{s+1} are open. For instance, for $t = 5$, we could have I_1, I_3 and I_5 open while I_2 and I_4 are both closed.

The face F of $P(G_n)$ induced by $x(T) \leq \alpha(T)$ is contained in some facet of $P(G_n)$, say that such a facet is induced by the valid inequality $\sum_{j=1}^{n} b_j x_j \leq \beta$. We shall prove that $(b_1, \ldots, b_m)$ is a positive multiple of χ^T. This is done by exploiting symmetries of the stable sets of cardinality k_s on I_s. Let $s \leq t$ and let $I_s = \{l, l+1, \ldots, r\}$. Let i satisfy (if any) $i, i+1 \in I_s$. Consider the two (possibly empty) intervals $T_1 = \{l, l+1, \ldots, i-3\}$ and $T_2 = \{i+4, i+5, \ldots, r\}$. One can check that $\alpha(T_1) + \alpha(T_2) = k_s - 1$. Furthermore, there is a stable set S in $T_1 \cup T_2$ with $|S| = k_s - 1$ such that $|S \cap \{l, r\}| \leq 1$, say $r \notin S$. Therefore we can augment S into a stable set in G_n with $|S \cap T| = \alpha(T) - 1$ by adding nodes to S such that suitable intervals become open and closed. This means that the incidence vectors of both $S \cup \{i\}$ and $S \cup \{i+1\}$ are roots of $x(T) \leq \alpha(T)$ and therefore $\sum_j b_j \chi_j^{S \cup \{i\}} = \sum_j b_j \chi_j^{S \cup \{i+1\}}$ which gives $b_i = b_{i+1}$. This implies that b_j has the same value, say β_s, for all $j \in I_s$.

Assume that $k \in I_s$ and $k+2 \in I_{s+1}$ (so $k+1 \notin T$). Choose a root S of $x(T) \leq \alpha(T)$ for which both I_s and I_{s+1} are open. Using the α-GREEDY algorithm we find that $\alpha(T \setminus \{k-2, k-1, \ldots, k+4\}) = \alpha(T) - 1$. Similar arguments as given above then gives that $b_k = b_{k+2}$, so $\beta_s = \beta_{s+1}$. Since s was arbitrary, we have shown that $(b_1, \ldots, b_n)$ is a multiple of χ^T, and therefore $\mathbf{x}(T) \leq \alpha(T)$ induces a facet of $P(G_n)$. ∎

When T is a 1-interval set we call the rank inequality $\mathbf{x}(T) \leq \alpha(T)$ a 1-*interval inequality*. Note that for the nonredundant 1-interval inequalities the value of $\alpha(T)$ is known, see Lemma 3.1.

4. Completeness. In this section we determine a complete and nonredundant linear description of the stable set polytope $P(G_n)$. Recall that each facet defining inequality $\mathbf{a}^T\mathbf{x} \leq \alpha$ which does not define a trivial facet must have nonnegative coefficients. The following result generalizes Lemma 3.2.

Lemma 4.1. *Let $\mathbf{a}^T\mathbf{x} \leq \alpha$ define a facet F_a of $P(G_n)$ which is not a trivial, clique or anti-web facet. Define $M = \max_j a_j$ and let $T = \{j \leq n : a_j = M\}$. Then the following statements hold:*

(i) T is a 1-interval set, say $T = I_1 + \ldots + I_t$, where $t \geq 2$ and

(ii) $|I_s| \equiv 1 \pmod 3$ for $s = 1, \ldots, t$.

Proof. (i) We first note that $T \neq V$ (otherwise $\mathbf{a}^T\mathbf{x} \leq \alpha$ would be equivalent to the anti-web inequality). We may then choose $i \in T$ such that $i - 1 \notin T$ and therefore $b_{i-1} < M$. Since F_a is not a clique facet there is a root S of F_a with $S \cap K = \emptyset$ where $K = \{i, i+1, i+2\}$ (otherwise F_a would be contained in the facet induced by the clique inequality $\mathbf{x}(K) \leq 1$). Thus there is an interval $I = \{l+1, \ldots, r-1\}$ satisfying $l, r \in S$, $K \subseteq I$ and $S \cap I = \emptyset$. We may assume that S is chosen such that $|I|$ is minimal.

We first observe that $l \in T$, i.e., $a_l = M$. For otherwise, $S' = (S \setminus \{l\}) \cup \{i\}$ would be a stable set whose incidence vector violates $\mathbf{a}^T\mathbf{x} \leq \alpha$. Therefore $l \neq i - 1$ as $i - 1 \notin T$. In fact we must have $l = i - 2$ for otherwise we could add the node i to S and violate the inequality $\mathbf{a}^T\mathbf{x} \leq \alpha$.

Thus we have shown that if $i \in T$ and $i - 1 \notin T$ then $i - 2 \in T$. This clearly implies that T is a 1-interval set $T = I_1 + \ldots + I_t$. If $t = 1$ then $T = V \setminus \{i\}$ for some i and it is easy to see that $\mathbf{a}^T\mathbf{x} \leq \alpha$ is implied by the anti-web inequality and the trivial inequality $-x_i \leq 0$. It follows that $t \geq 2$, and Property (i) holds.

(ii) Assume that $|I_s| \equiv 2 \pmod 3$ for some $s \leq t$, and let $I_s = \{l, \ldots, r\}$. Let S be a root of F_a with $l - 1 \in S$ (such a root exists, otherwise F_a would be contained in the hyperplane given by $x_{l-1} = 0$). This implies that S also contains the set

$\{j \in I_s : j \equiv l-1 \bmod (3)\} \cup \{r\}$ (i.e., nodes $l+2, l+5, \ldots$ lying in T plus r). Otherwise the distance between two consecutive nodes in S would be at least 4, and we could modify S by replacing $l-1$ by l, $l+2$ by $l+3$ etc. This produces a stable set violating $\mathbf{a}^T\mathbf{x} \leq \alpha$ because $a_{l-1} < a_l$. Thus each root containing $l-1$ also contains $r+1$. Due to symmetry, we conclude that a root of F_a contains $l-1$ if and only if it contains $r+1$. But this means that F_a is contained in the hyperplane given by $x_{l-1} - x_{r+1} = 0$; contradicting that F_a is a facet. This proves that $|I_s| \not\equiv 2 \pmod 3$ for all $s \leq t$.

Assume that $|I_s| \equiv 0 \pmod 3$ for some $s \leq t$, say $|I_s| = 3k$. Let $I_s = \{l, \ldots, r\}$. We observe, using similar arguments to those of the previous paragraph, that for each root S of F_a we have (a) S contain at most one of the nodes $l-1$ and $r+1$, and (b) if S contains either $l-1$ or $r+1$ then $|S \cap I_s| = k$.

Furthermore, as $|I_s| = 3k$, $\mathbf{x}(I_s) \leq k$ is a valid inequality for $P(G_n)$ obtained by adding k clique inequalities for nodes in I_s ($x_i + x_{i+1} + x_{i+2} \leq 1$, $x_{i+3} + x_{i+4} + x_{i+5} \leq 1$ etc.) Therefore there must be a root S of F_a satisfying $\mathbf{x}(I_s) \leq k$ with strict inequality, i.e., $|S \cap I_s| \leq k-1$. This implies, due to the observation above, that $|S \cap \{l-1, \ldots, r+1\}| = k$. Let S' be the set obtained from S by replacing the $k-1$ nodes in $S \cap I_s$ by the k nodes $l+1, \ldots, r-1$. Then S' is a stable set which violates $\mathbf{a}^T\mathbf{x} \leq \alpha$ (by the amount of M). This proves that $|I_s| \not\equiv 0 \pmod 3$ for all $s \leq t$ and the proof is complete.

□

The next result concerns projection of facets. It gives a simple procedure for producing facets for $P(G_{n-3})$ from those of $P(G_n)$. The technique has some resemblance to a shrinking result given in [1].

Consider an inequality

$$\sum_{j=1}^{n} a_j x_j \leq \alpha \tag{4.1}$$

which defines a facet F_a of $P(G_n)$ which is different from each trivial, clique or anti-web facet. (The procedure also works for the anti-web facet, but this is not of importance here). Let, as before, $M = \max_j a_j$ and $T = \{j \leq n : a_j = M\}$. From Lemma 4.1 we have that T is a 1-interval set $T = I_1 + \ldots + I_t$ with $t \geq 2$ and $|I_s| \equiv 1 \pmod 3$ for each s. Consider the interval I_s and let $|I_s| = 3k+1$. Our procedure may be applied whenever $k \geq 1$. Assume, for notational simplicity, that $I_s = \{n-3k, \ldots, n\}$

so, in particular, $1 \notin T$. We then have the following result:

LEMMA 4.2. *The inequality*

$$\sum_{j=1}^{n-3} a_j x_j \leq \alpha - M \tag{4.2}$$

is valid for $P(G_{n-3})$. *Moreover it defines a facet of* $P(G_{n-3})$.

Proof. Assume that there is a stable set S in G_{n-3} with $\sum_{j=1}^{n-3} a_j \chi_j^S > \alpha - M$. Consider first the case when $n-3 \notin S$. Then S can not contain both the nodes 1 and $n-4$ (as S is stable), say that $1 \notin S$. Then $S' = S \cup \{n-1\}$ is a stable set in G_n and $\sum_{j=1}^{n} a_j \chi_j^{S'} > \alpha - M + M = \alpha$ which contradicts that (4.1) is valid for $P(G_n)$. Consider the remaining case when $n-3 \in S$. Then $1, 2 \notin S$ and therefore $S \cup \{n\}$ is a stable set in G_n and its incidence vector violates (4.1). It follows, by contradiction, that (4.2) is valid for $P(G_{n-3})$.

Assume that there is a root S of (4.1) with $S \cap \{n-3, n-2, n-1, n\} = \emptyset$. If $1 \in S$ we could violate (4.1) by replacing 1 by n, so we conclude that $1 \notin S$. But this is also impossible, for then we could add the node $n-1$ and violate (4.1). Thus every root of (4.1) contains either one or two nodes in the set $\{n-3, n-2, n-1, n\}$.

Since (4.1) defines a facet of $P(G_n)$ there is a nonsingular matrix $\mathbf{B} \in \mathbb{R}^{n,n}$ with rows being the incidence vector of stable sets that are roots of (4.1) (as $\mathbf{0}$ is not a root of the inequality, the affine rank and the linear rank of the roots coincide). The columns of $\mathbf{B} = (b_{i,j})$ correspond to the nodes $1, \ldots, n$. As shown above each row of $\mathbf{B}$ contains one or two 1's in positions $n-3, n-2, n-1, n$ and, after a reordering of rows, we may assume that all the rows with two 1's in the mentioned positions are the last rows of $\mathbf{B}$. Let $\mathbf{B}' = (b'_{i,j}) \in \mathbb{R}^{n,n-3}$ be the matrix obtained by replacing the last four columns of $\mathbf{B}$ by a single column with ith entry being 1 if $\sum_{j=n-3}^{n} b_{i,j}$ equals 2 and 0 otherwise. By this construction it is clear that each row of $\mathbf{B}'$ is the incidence vector of a stable set in G_{n-3}.

We claim that $\text{rank}(\mathbf{B}') = n-3$. To prove this, let $\mathbf{b}_1, \ldots, \mathbf{b}_{n-4} \in \mathbb{R}^n$ be the first $n-4$ columns of $\mathbf{B}'$ and $\mathbf{B}$ (these columns are equal in the two matrices). These vectors are linearly independent as $\mathbf{B}$ is nonsingular. Assume that the last column $\mathbf{b}^T = \begin{bmatrix} \mathbf{0}^T, \mathbf{1}^T \end{bmatrix}$ of $\mathbf{B}'$ lies in the span L of the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_{n-4}$. Then $-\mathbf{b}$ also lies in L and there is an $\mathbf{x}' \in \mathbb{R}^{n-4}$ such that

$$\begin{bmatrix} \mathbf{a}_1, \ldots, \mathbf{a}_{n-4} \end{bmatrix} \mathbf{x}' = \begin{bmatrix} \mathbf{0} \\ -\mathbf{1} \end{bmatrix}.$$

Thus, with $\mathbf{x}^T = \left[\ (\mathbf{x}')^T, 1, 1, 1 \ \right]$, we obtain that

$$\mathbf{Bx} = \begin{bmatrix} \mathbf{0} \\ -\mathbf{1} \end{bmatrix} + \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix} = \mathbf{1}$$

due to the structure of the last four columns of $\mathbf{B}$. On the other hand the rows of $\mathbf{A}$ are the incidence vectors of roots of $\mathbf{a}^T\mathbf{x} \leq \alpha$, so $\mathbf{Ba} = \alpha\mathbf{1}$. Since $\mathbf{B}$ is nonsingular we conclude that $\alpha\mathbf{x} = \mathbf{b}$ and therefore $a_j = \alpha$ for $n-3 \leq j \leq n$. Thus the inequality $\mathbf{a}^T\mathbf{x} \leq \alpha$ has the form

$$\sum_{j=1}^{n-4} a_j x_j + \alpha \sum_{j=n-3}^{n} x_j \leq \alpha.$$

Inserting the stable set χ^{n-1} we conclude that $a_j = 0$ for $2 \leq j \leq n-4$. This contradicts the form of $\mathbf{a}^T\mathbf{x} \leq \alpha$ where $T = I_1 + \ldots + I_t$ with $t \geq 2$ and a_j positive (and maximal) for each $j \in T$. This proves that the columns of $\mathbf{B}'$ are linearly independent and it follows that $\mathbf{B}'$ has rank $n-3$. Thus there are $n-3$ linearly independent rows of $\mathbf{B}'$ and since all these are roots of (4.2) we have shown that (4.2) defines a facet of $P(G_{n-3})$.

□

Theorem 4.3. *For each n the stable set polytope $P(G_n)$ is the solution set of the nonnegativity constraints, the clique inequalities, the anti-web inequality (3.4) and the nonredundant* 1*-interval inequalities described in Theorem 3.3.*

Proof. Let $\mathbf{a}^T\mathbf{x} \leq \alpha$ be a facet defining inequality for $P(G_n)$ which is neither a nonnegativity constraint, a clique inequality or the anti-web constraint. We shall prove that the inequality is a positive multiple of some 1-interval inequality.

Due to Lemma 4.1 the inequality $\mathbf{a}^T\mathbf{x} \leq \alpha$ may be written as

$$M \sum_{j \in T} x_j + \sum_{j \notin T} a_j x_j \leq \alpha \tag{4.3}$$

where $T = I_1 + \ldots + I_t$ is a 1-interval set and $M = \max_j a_j$. Recall, also from Lemma 4.1, that $|I_s| \equiv 1 \pmod 3$ for $s = 1, \ldots, t$. By repeated application of the reduction procedure of Lemma 4.2, say p times, we get an inequality

$$M \sum_{j \in T'} x_j + \sum_{j \notin T'} a_j x_j \leq \alpha - pM \tag{4.4}$$

which defines a facet of $P(G_{n-3p})$. Here T' is a 1-interval set for which each interval consists of exactly one node. Thus, the coefficients in this inequality alternates between M and numbers strictly smaller than M. We show that all these numbers different from M are equal to 0.

Let $i \in T'$. Since (4.4) is not a clique facet, there is a root S of this inequality with $S \cap \{i, i+1, i+2\} = \emptyset$. Note that $a_{i+1} < a_i = a_{i+2} = M$ as T' is a 1-interval set. S can not contain $i-1$ or $i+3$ (both outside T') for then we could modify S by replacing that node by i or $i+2$ and violate (4.4). Thus $S \cap \{i-1, i, i+1, i+2, i+3\} = \emptyset$ which implies that $S \cup \{i+1\}$ is a stable set. But since S is a root, we see that $a_{i+1} = 0$. This shows that $a_i = 0$ for each $i \notin T'$ and therefore (4.4) is a positive multiple of a 1-interval inequality with all intervals of length 1 (the right hand side must have the proper value otherwise the inequality would be redundant). This also proves that the original inequality (4.3) is M times a 1-interval inequality (in particular, $\alpha = M\alpha(T)$) and the theorem follows. □

Examples. When $n = 9$ the a minimal linear system for $P(G_9)$ consists of nonnegativity and clique constraints as well as the following 1-interval inequalities

$$
\begin{array}{cccccccccc}
 & x2 & & +x4 & & +x6 & +x7 & +x8 & +x9 & \leq 2 \\
 & x2 & & +x4 & +x5 & +x6 & +x7 & & +x9 & \leq 2 \\
 & x2 & +x3 & +x4 & +x5 & & +x7 & & +x9 & \leq 2 \\
x1 & & +x3 & & +x5 & & +x7 & +x8 & +x9 & \leq 2 \\
x1 & & +x3 & & +x5 & +x6 & +x7 & +x8 & & \leq 2 \\
x1 & & +x3 & +x4 & +x5 & +x6 & & +x8 & & \leq 2 \\
x1 & +x2 & & +x4 & & +x6 & & +x8 & +x9 & \leq 2 \\
x1 & +x2 & +x3 & & +x5 & & +x7 & & +x9 & \leq 2 \\
x1 & +x2 & +x3 & +x4 & & +x6 & & +x8 & & \leq 2.
\end{array}
$$

These inequalities correspond to 1-interval sets with three intervals of cardinalities 4, 1 and 1. Consider next $n = 14$. Then the anti-web inequality $\mathbf{x}(V) \leq 5$ defines a facet of $P(G_{14})$. The inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \quad + x_9 \quad + x_{11} \quad + x_{13} \quad + x_{15} \leq 4$$

is the 1-interval inequality for $T = I_1 + \ldots + I_5$ where $I_1 = \{1, \ldots, 7\}$, $I_2 = \{9\}$, $I_3 = \{11\}$, $I_4 = \{13\}$, $I_5 = \{15\}$. It defines a facet of $P(G_{14})$. Similarly, the 1-interval $T = I_1 + \ldots + I_5$ with $I_1 = \{1, 2, 3, 4\}$, $I_2 = \{6, 7, 8, 9\}$, $I_3 = \{11\}$, $I_4 = \{13\}$, $I_5 = \{15\}$ gives the inequality

$$x_1 + x_2 + x_3 + x_4 \quad + x_6 + x_7 + x_8 + x_9 \quad + x_{11} \quad + x_{13} \quad + x_{15} \leq 4$$

In fact, the 1-interval sets that correspond to facets of $P(G_{16})$ all consist of $t = 5$ intervals with cardinalities either $7, 1, 1, 1, 1$ or $4, 4, 1, 1, 1$. The minimal linear system for $P(G_{16})$ consists of 48 inequalities corresponding to 1-interval sets in addition to the anti-web inequality, 16 clique inequalities and 16 nonnegativity constraints.

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