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Abstract

The analysis of chaotic time series requires proper reconstruction of the state space from the available data in order to successfully estimate invariant properties of the embedded attractor. Using the correlation dimension, we discuss the applicability of the two most common methods of reconstruction, the method of delays (MOD) and the Singular Spectrum Approach (SSA). Contrary to previous discussions, we found that the two methods perform equivalently in practice for noise-free data provided the parameters of the two methods are properly related. In fact, the quality of the reconstruction is in both cases determined by the choice of the time window length τ_w and is independent of the selected method. However, when the data are noisy, we find that SSA outperforms MOD.

1 Introduction

State space reconstruction is the first step in non-linear time series analysis including estimation of invariants and prediction and consists of viewing a time series $x_k = x(k\tau_s)$, for $k = 1, \dots, N$, where τ_s is the sampling time, in Euclidean space $\mathbb{R}^m$. (For a review on these topics see [11], [16], [18] and [2].) Takens [30] showed that theoretically the *embedding dimension* m should satisfy $m \geq 2\lceil d \rceil + 1$, where d is the fractal dimension and $\lceil d \rceil$ is the lowest integer greater than d, in order to preserve the dynamical properties of the original attractor.

Two popular methods of reconstruction are MOD (Method Of Delays) and SSA (Singular Spectrum Approach). They are theoretically equivalent [28], [4] but may differ in practice with limited amounts of possibly noisy data. Both approaches have been extensively investigated and used in applications and each has its proponents (for MOD see for example [22], [2] and for SSA see [20] [31], [25] and [29]). Considering the correlation dimension, we show that these methods give similar results also in practice under noise-free conditions with properly chosen parameter values. From the comparisons, we conclude that the key in reconstruction with either MOD or SSA is to use the same time window τ_w covered by the embedding vectors [14].

The two methods are briefly presented in Section 2. In Section 3, we discuss how to achieve optimal reconstructions when the time series is generated by a continuous system and compare the two methods for this type of data. In Section 4, data from discrete systems are treated and the conclusions are presented in Section 5.

2 Methods of reconstruction

We review briefly the reconstruction of an attractor in $\mathbb{R}^m$ with MOD and SSA:

MOD: The m-dimensional reconstructed state vector is

$$\mathbf{x}_k^m = [x_k, x_{k+\rho}, \ldots, x_{k+(m-1)\rho}]^T \quad (1)$$

where ρ is a multiple integer of τ_s so that the *delay time* τ is defined as $\tau = \rho\tau_s$ [23]. The m coordinates are samples (separated by a fixed τ) from a time window length τ_w, such that $\tau_w = (m-1)\tau$. We use MOD(τ,m) or MOD(ρ,m) to emphasize the two parameters.

SSA: A "large" p-dimensional state vector is first derived from successive samples as $\mathbf{x}_k^p = [x_k, x_{k+1}, \ldots, x_{k+p-1}]^T$ which can be seen as reconstruction with MOD(1,p). The final m-dimensional state vector $\mathbf{x}_k^m$ is a projection onto the first m principle components defined by the data in $\mathbb{R}^p$ using Singular Value Decomposition (SVD), i.e. $\mathbf{x}_k^m = \mathbf{P}\mathbf{x}_k^p$, where $\mathbf{P}$ is an $m \times p$ matrix [6]. We use the notation SSA(p,m) and have $\tau_w = (p-1)\tau_s$. Note that τ_w and m are the same for the two methods. Actually these two parameters are common to any method of reconstruction.

We can extend the definition of SSA and consider $\tau > \tau_s$ when constructing the initial high dimensional vectors. Keeping again τ_w fixed, we allow combinations of τ and p, such that $\tau_w = (p-1)\tau$ for the initial embedding. In that case, the coordinates of the final embedding in $\mathbb{R}^m$ with SSA are restricted to be linear combinations of fewer measurements from τ_w than when $\tau = \tau_s$.

The difference betwen the two methods is that in MOD the m coordinates are samples seperated by a fixed τ and cover a time window length τ_w while in the standard SSA all the available samples in τ_w are initially used, and they are further processed with SVD so that the final m coordinates are linear combinations of these measurements. In this work, we investigate which of these two ways of passing information from τ_w to the point representation $\mathbf{x}_k^m$ is the best. Certainly, there are many other schemes (differentiating, weighting or averaging the samples in τ_w, see [23], [7], [4] and [27]) but since MOD and SSA are the dominant methods we will confine ourselves to them.

It seems that most methodologists who have explored the issue of state space reconstruction have spent little effort on the proper choice of τ_w, while practitioners have chosen τ_w arbitrarily or indirectly, e.g. when using MOD they find m and τ indepedently from one of the many existing methods. Concerning the selection of τ_w, we suggest as a lower limit the *mean orbital period* τ_p, which operationally can often be estimated as the average of time differences between peaks of the oscillations of the original or filtered time series. For a detailed discussion of this topic we refer to [14]. However, in the simulations below we use a broad range of values for the parameters τ_w (and thus p), τ and m in order to assess the performance of MOD and SSA.

3 Reconstructions for continuous systems

MOD and SSA are evaluated using the correlation dimension ν, a measure related to the geometry of the attractor. To estimate ν, first the correlation integral $C(l)$ is computed

$$C(l) = \frac{1}{N(N-1)} \sum_{i,j=1,|i-j|>K}^{N} \Theta(l - ||\mathbf{x}_i - \mathbf{x}_j||) \quad (2)$$

which gives for each distance l the average fraction of the number of points with inter-distances less than l [10]. The function $\Theta(x)$ is the Heaviside function ($\Theta(x) = 0$ when $x < 0$ and $\Theta(x) = 1$ when $x \geq 0$). The inter-distances are measured with the maximum norm. Points that are temporally closer than K are omitted in the computations. The ν is estimated from the slope (scaling) of the graph of $\log C(l)$ vs $\log l$ for a sufficiently large interval of small l distances. To assure a good estimation the same ν value should be found for different reconstructions with systematically varying parameters.

Before comparing the two methods, the role of m in SSA has to be clarified. For SSA, the free parameter m is not critical at all and any choice over a lower limit would give essentially the same reconstruction because the additional coordinates, correspond to less significant singular values and give negligible variance assuming τ_w is sufficiently large. For computational purposes we still want to find a lower limit for m. This limit can be easily identified if we estimate an invariant, such as the correlation dimension ν, in successively higher spaces [14]. In Fig. 1, we show the estimation of ν for the Lorenz attractor [19] reconstructed with SSA(75,m) where $m = 2, \ldots, 10$ ($p = 75$ corresponds to the time window length determined by τ_p, estimated from the pseudo-periodic orbits in each of the two loops of the Lorenz attractor). We observe

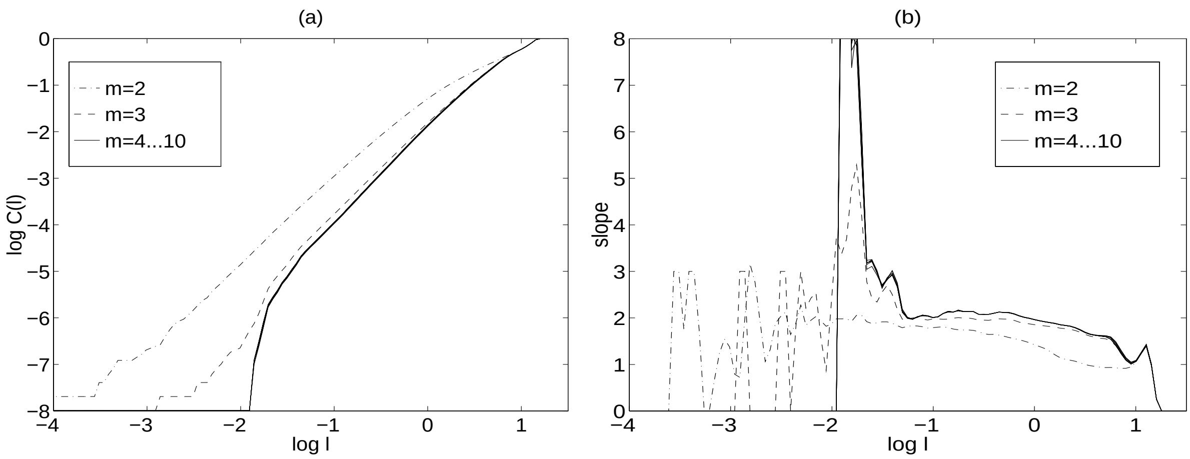

Figure 1: Correlation dimension estimation with SSA for a time series of 10000 measurements of the x-variable of the Lorenz system sampled with $\tau_s = 0.01$sec. (a) Log-log plot of the correlation integral $C(l)$ versus the interdistance l for reconstructions with SSA(75,m) where $m = 2, \ldots, 10$. (b) Plot of the slopes of the correlation integrals in (a).

from Fig. 1 the saturation of $C(l)$ (and ν) for $m = 4$. Obviously, increasing m beyond 4 has no effect on the estimation of ν for this selection of τ_w. (It turns out that $m = 3$ is not sufficient to estimate ν for the Lorenz system for the data size we used here.)

Comparing the two methods we find with MOD that any combination of τ and m (over some limit value) that satisfies $\tau_w = (m-1)\tau$ is sufficient, which rules out the search for τ that assures uncorrelated and orthogonal coordinates [14]. The same is observed when SSA is used instead (following the extended definition involving τ). The quality of the reconstruction does not change essentially as long as τ_w is the same as we show in Table 1, where we report the estimated correlation dimension $\overline{\nu}$ for all possible reconstructions with MOD and SSA for the Lorenz data when $\tau_w \simeq 0.75$sec.

All estimates of ν in this paper are taken to have the least variance in a scaling region

MOD				SSA			
m	τ	$\overline{\nu}$	**sd**	**p**	τ	$\overline{\nu}$	**sd**
75	1	2.07	0.045	75	1	2.08	0.032
38	2	2.05	0.031	38	2	2.08	0.026
25	3	2.05	0.033	25	3	2.06	0.030
19	4	2.05	0.033	19	4	2.05	0.041
15	5	2.05	0.037	15	5	2.07	0.029
12	6	2.04	0.048	12	6	2.09	0.035
10	7	2.08	0.049	10	7	2.08	0.037
9	8	2.08	0.047	9	8	2.08	0.030
8	9	2.04	0.043	8	9	2.07	0.031
7	10	2.03	0.045	7	10	2.06	0.029
6	12	2.07	0.045	6	12	2.07	0.045

Table 1: Estimates of ν for the Lorenz attractor with standard deviation (sd) for $\tau_w \simeq 75\tau_s$ ($\tau_s = 0.01$sec) and different parameters of MOD and SSA. For SSA, $m = 5$ for all combinations of τ and p. The correct ν is 2.06.

of inter-distances $[l_0, l_1]$ such that $l_1/l_0 \geq 4$. The results show the equivalence of all these reconstructions. We have found the same equivalence for other selections of τ_w.

In the estimation of the correlation dimension ν with MOD, one traditionally keeps τ fixed and increases m, which means that we increase τ_w with a time length equal to τ each time we increase the dimension m with one. In this way, the slope curves do not become identical for larger m (as it does for SSA for a given τ_w or p, see Fig. 1), because τ_w varies with m. However, we hope to observe saturation of the slope of the correlation integral over some region of the inter-distances l for a range of m dependent on the selected τ (see Fig(2a)). The equivalent process with SSA is done by increasing p instead (see Fig. 2b). There is a perfect matching of the slopes obtained by MOD and SSA in Fig. 2 corresponding to the same τ_w.

For a given working embedding dimension m, the variation of τ_w in reconstruction implies a change of τ when we apply MOD and of p when we apply SSA. In Fig. 3, we show the estimate of ν in the projected space $\mathbb{R}^5$ as a function of τ_w, increasing τ for MOD and increasing p for SSA. Both MOD and SSA give bad reconstructions when τ_w is very small (significantly less than τ_p), which results in bad estimates of ν. We stress that this is due to limited data. For larger time series better estimation of ν would be obtained for small τ_w in accordance with Takens' theorem. On the other hand, an upper limit for τ_w cannot readily be delineated and it seems that once τ_w reaches τ_p, additional measurements do not affect the quality of reconstruction. For short time series it is observed that the results diverge for time windows significantly larger than τ_p [14].

We have done the same tests on data from other systems and found similar results. The lower value of τ_w that gives "good" reconstructions is always at the level of τ_p. The reconstruction is successful with either MOD or SSA and we can then get confident estimates of ν if the time series is sufficiently large. Some results are reported in Table 2. In line three, the four dimensional Rössler system (the so-called Rössler hyperchaos) is indexed with 4D to distinguish it from the three dimensional Rössler system in line one. In line four of Table 2 we give results for the 2-torus used in [8]

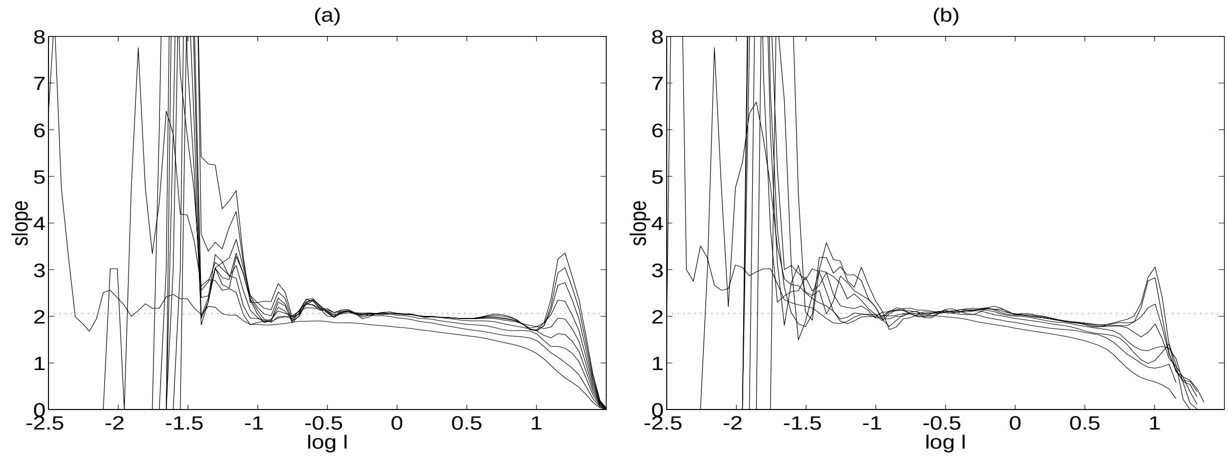

Figure 2: Correlation dimension estimation for varying τ_w using MOD in (a) and SSA in (b) on the Lorenz data ($N = 10000$, $\tau_s = 0.01$sec). The horizontal dotted line in both figures shows the correct plateau of $\nu = 2.06$. In (a) the τ is fixed to the minimum of mutual information [9] ($\tau = 18\tau_s$) whereas τ_w increases with m, $m = 2, \ldots, 10$. The slope curves are plotted beginning with the lowest for the smallest m in the interval $[-1, 1.5]$ of $\log l$. In (b) the corresponding increase of τ_w is done by increasing p while m is chosen as in (a). The slope curves are displayed in the same way as in (a).

where SSA was claimed to be deficient. We believe that the conclusion in [8] was misleading because a too small p was chosen. The results in Table 2 show that MOD and SSA are approximately equivalent.

In the presense of noise in the data, it turns out that the in-built filter in the SSA reconstruction makes SSA superior to MOD. Geometrically, the filtering effect of SSA lies in the SVD-transformation of the data in $\mathbf{R}^p$ prior to the projection from p to m dimensions. The first coordinate axes of the new basis of $\mathbf{R}^p$ defined by SVD, have the largest variation while the last coordinates mainly express noise. Discarding the directions of little data variation we actually filter out noise. The in-built filter of SSA comprises an important advantage of SSA over MOD and establishes the applicability of SSA to "real" data [1]. In Table 3, we show results for the simulated data corrupted with 5% noise and for three experimental data sets. We see that SSA gives more confident and unbiased estimates of ν than MOD. For the Taylor Couette data in chaotic

					MOD			SSA			
system	**data size**	τ_s	τ_w	**m**	τ	$\overline{\nu}$	**sd**	**p**	$\overline{\nu}$	**sd**	ν
Rössler 3D	10000	0.10	60	5	15	1.95	0.031	61	1.95	0.033	2.01
Rabinovich [24]	10000	0.10	28	5	7	2.21	0.060	29	2.08	0.038	2.19
Rössler 4D [26]	10000	0.10	60	6	12	2.71	0.117	61	2.96	0.118	3.01
Torus Fraser [8]	10000	0.14	40	5	10	1.90	0.056	41	2.10	0.107	2.00

Table 2: Estimates of ν with standard deviation (sd) for data from different systems using MOD (τ,m) and SSA (p,m) for specified p, τ and m. We use $\tau_w \simeq \tau_p$ expressed in units of τ_s in column 4 and keep the same m for MOD and SSA. At the last column we quote the correct correlation dimensions for comparison.

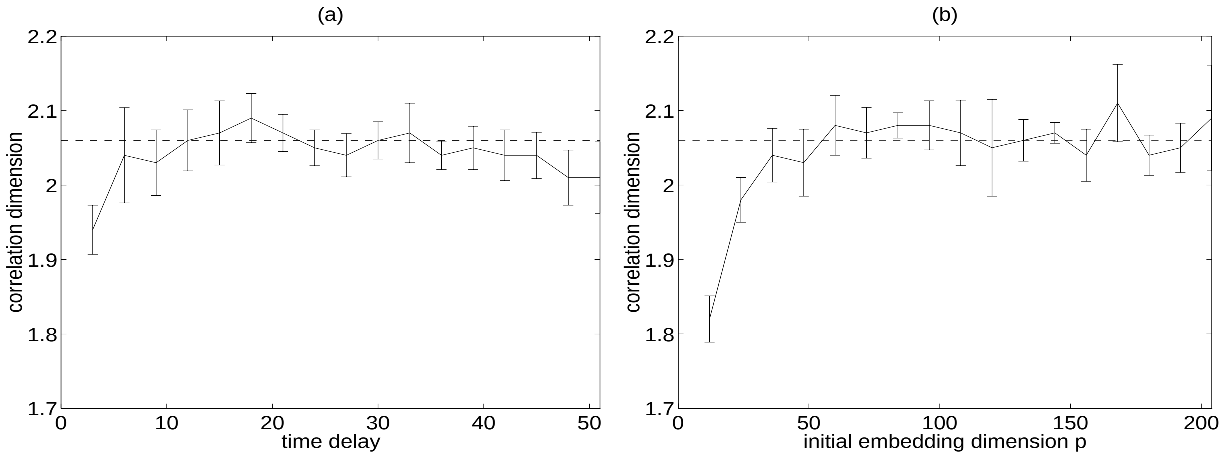

Figure 3: Correlation dimension estimation with MOD(τ,5) and SSA(p,5) for the Lorenz data ($N = 10000$, $\tau_s = 0.01$sec). (a) Plot of the correlation dimension estimate $\overline{\nu}$ for MOD reconstruction with different τ and for $m = 5$. (b) Plot of $\overline{\nu}$ for SSA reconstruction with different p and for $m = 5$. The bars indicate the standard deviation of the estimate and the horizontal stippled line shows the correct plateau of $\nu = 2.06$.

regime and the Belousov data (at line 6 and 8 in Table 3, respectively), true ν values are not provided. For the Belousov data, the ν estimates from MOD and SSA match very well. For the Taylor Couette data, SSA gives somehow smaller estimate for ν than MOD which agrees with the ν estimate obtained with the correction scheme in [15].

					MOD			SSA			
system	**data size**	$\tau_\mathbf{s}$	$\tau_\mathbf{w}$	**m**	τ	$\overline{\nu}$	**sd**	**p**	$\overline{\nu}$	**sd**	ν
Lorenz	10000	0.01	72	5	18	1.90	0.120	73	2.09	0.090	2.06
Rössler 3D	10000	0.10	60	5	15	2.14	0.092	61	1.97	0.032	2.01
Rabinovich	10000	0.10	28	5	7	2.39	0.095	29	2.22	0.049	2.19
Rössler 4D	10000	0.10	60	6	12	2.79	0.243	61	2.99	0.201	3.01
Torus Fraser	10000	0.14	40	5	10	2.46	0.290	41	2.22	0.066	2.00
Taylor Chaos [3]	16384		50	6	10	2.61	0.171	51	2.50	0.150	
Taylor Periodic	16384		40	5	10	1.03	0.028	41	1.01	0.006	1.00
Belousov [17]	16384		35	6	7	1.46	0.081	36	1.45	0.048	

Table 3: Estimates of ν with standard deviation (sd) for data from different systems corrupted with 5% noise and for experimental data using MOD (τ,m) and SSA (p,m) as in Table 2.

4 Reconstructions for discrete systems

Successive measurements from discrete chaotic processes are typically linearly uncorrelated. We can easily observe this from the singular spectrum given by SVD (see Fig. 4). The singular values lie at the same level similarly to white noise (as for the logistic map in Fig. 4a) or decrease comparably slowly (as for the Henon map [12] in

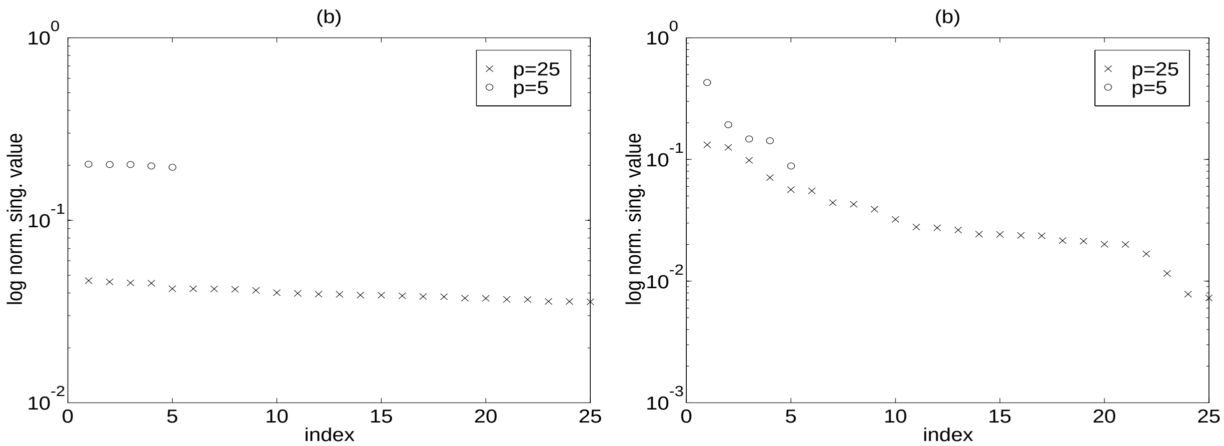

Figure 4: (a) Semilog plot of the normalized singular spectrum for two reconstructions with initial embedding dimensions $p = 5$ and $p = 25$ for the one-dimensional logistic map, $x_k + 1 = 4x_k(1 - x_k)$. (b) The same for the Henon map [12].

Fig. 4b). For the maps we set $\tau_s = 1$.

Any map can be seen as a Poincarè map, i.e. defined on a Poincarè section drawn for an attractor generated by a flow in a state space of dimension one higher than the dimension of the Poincarè section. Hence, successive points generated by a map can be seen as generated by a flow every orbital period. Thus for measurements from maps, as well as from continuous systems with large sampling time τ_s, it seems advantageous to fix the parameters $\tau = 1$ and $p = m$ when reconstructing with MOD and SSA, respectively. In any case, this leaves only one parameter to be adjusted because $\tau_w = (m - 1)$. This indicates the inappropriateness of using SSA here since there is no need for projection from p to m dimensions.

However, in order to show the equivalence of the two methods also for this type of data, we consider reconstructions with $\tau > 1$ for MOD and $p > m$ for SSA. When $\tau_w > m - 1$, the macroscopic form of the attractor gets distorted and the fractal structure can be observed only on small scales. For the estimation of the correlation dimension this means that the scaling region gets smaller and may even be masked when τ_w is too large for the given data size. This holds when either MOD or SSA is used as shown in Fig. 5. Note the breaking of the scaling at large distance scales with the increase of τ_w (for $\log l$ around -1 in Fig. 5a, $\log l \in [-2, -1]$ in Fig. 5b and $\log l \in [-4, -1]$ in Fig. 5c). Obviously, the increase of the data size allows the observation of the fractal structure of the attractor on smaller scales. For $\tau_w = 2$ and $\tau_w = 4$ in Fig. 5a and Fig. 5b, respectively, the scaling interval extends to smaller distances when the time series length is increased from 2000 to 30000 but for $\tau_w = 9$ in Fig. 5c, even 30000 data are not enough to give clear scaling. However, it is well-known that for infinite noise-free data, any τ (or p) is appropriate as long as $m \geq 2\lceil d \rceil + 1$, i.e. the insufficiency of reconstruction is solely due to the limited or corrupted data. The equivalence in the performance of MOD(τ,2) and SSA(p,2), i.e. under the same τ_w, shown in Fig. 5 holds also for other choices of m.

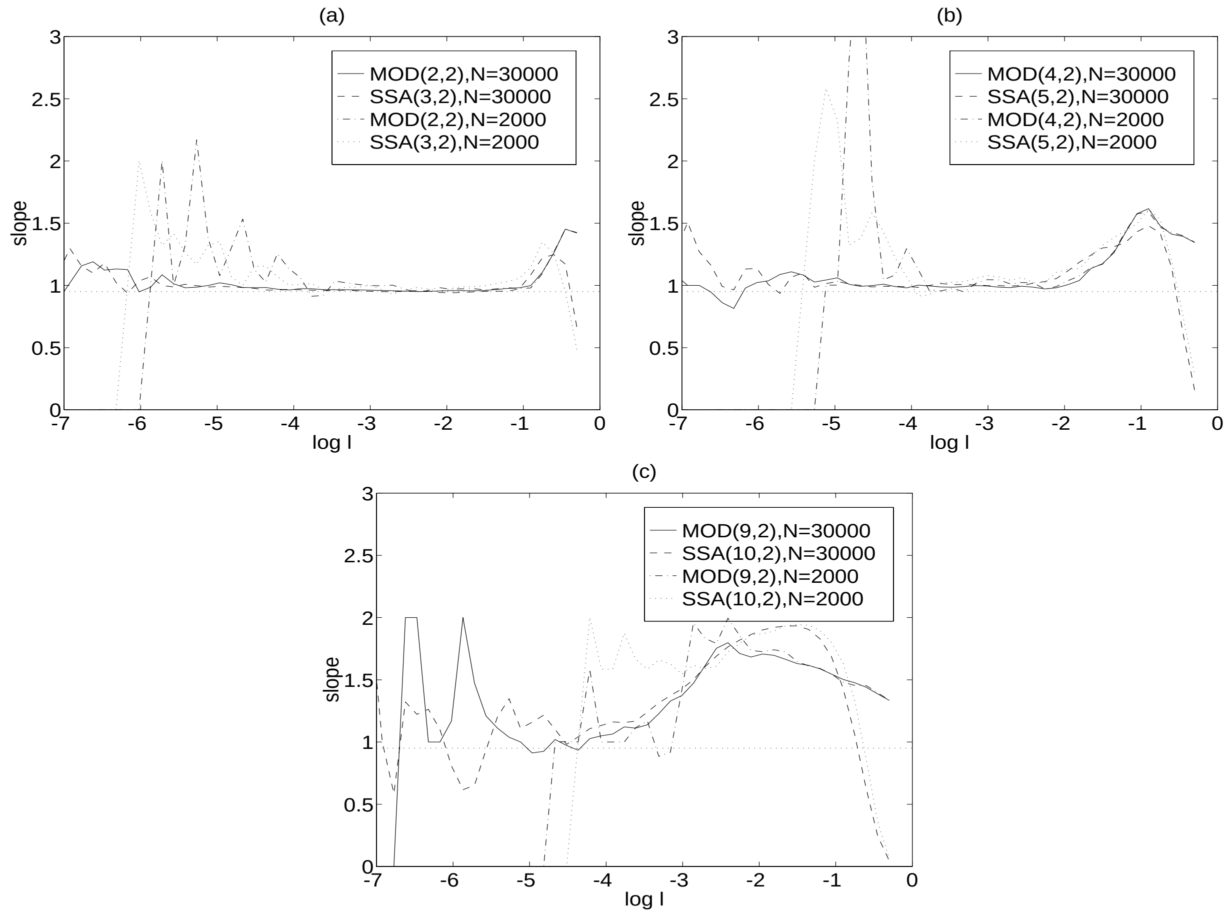

Figure 5: Estimate of the correlation dimension from reconstructions with $\tau_w > m$ for data from the logistic map. In each plot the slope of the correlation integral is shown as a function of the $\log$ of interdistances l for four different reconstructions (with MOD and SSA and noise-free time series of length 2000 and 30000) as explained in the legend. In (a) $\tau_w = 2$ and $m = 2$, in (b) $\tau_w = 4$ and $m = 2$, and in (c) $\tau_w = 9$ and $m = 2$.

5 Conclusions

Some misunderstanding has surrounded the use of SVD in the literature on state space reconstruction. This is partly due to the selection of a too short τ_w when implementing SSA (e.g. see [8]) and partly due to the misleading attempt of finding the proper embedding dimension from the cut-off of the singular spectrum (e.g. see the comments in [5] and [21] and the application in [29]). Disregarding these two improper setups, SSA turns out to be a legitimate and useful method for reconstruction.

For noise-free and limited data the equivalence of MOD and SSA as reconstruction methods is demonstrated, provided the time window τ_w is kept the same. In particular, using the estimation of the correlation dimension, we found that the results from MOD and SSA coincide for all reconstruction set-ups we tested under the same τ_w. For noisy data, SSA performs better than MOD probably due to the in-built filter property of SSA. Since the existing methods for non-linear filtering demand long time series (e.g. see [13]) SSA is particularly important in the reconstruction from short and noisy time series.

The critical parameter that determines the quality of the reconstruction is τ_w. For data from discrete systems, this is equal to $m-1$. For data from continuous systems, we suggest generally $\tau_w \geq \tau_p$ where τ_p is the mean orbital period. Smaller values for τ_w reduce the computational demands and $\tau_w \simeq \tau_p$ thus provides a reasonable starting point [14].

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References

[1] T. Aasen, D. Kugiumtzis, and S. H. G. Nordahl. Procedure for estimating the correlation dimension of optokinetic nystagmus signal. *Computers and Biomedical Research*, in press, 1997.

[2] H. D. I. Abarbanel, R. Brown, J. J. Sidorowich, and L. S. Tsimring. Analysis of observed chaotic data in physical systems. *Reviews of Modern Physics*, 65(4):1331 – 1392, 1993.

[3] A. Brandstater and H. Swinney. Strange attractors in weakly turbulent Couette-Taylor flow. *Physical Review A*, 35:2207 – 2220, 1987.

[4] D. S. Broomhead, J. P. Huke, and M. R. Muldoon. Linear filters and non-linear systems. *Journal of the Royal Statistical Society Series B-Methodological*, 55(2):373 – 382, 1992.

[5] D. S. Broomhead, R. Jones, and G. P. King. Comment on "singular-value decomposition and embedding dimension". *Physical Review A*, 37(12):5004 – 5005, 1988.

[6] D. S. Broomhead and G. P. King. Extracting qualitative dynamics from experimental data. *Physica D*, 20:217 – 236, 1986.

[7] J. D. Farmer and J. J. Sidorowich. Exploiting chaos to predict the future and reduce noise. In Y. C. Lee, editor, *Evolution, Learning and Cognition*, pages 277 – 330. World Scientific, 1988.

[8] A. M. Fraser. Reconstructing attractors from scalar time series: a comparison of singular system and redundancy criteria. *Physica D*, 34:391 – 404, 1989.

[9] A. M. Fraser and H. Swinney. Independent coordinates for strange attractors from mutual information. *Physical Review A*, 33:1134 – 1140, 1986.

[10] P. Grassberger and I. Procaccia. Measuring the strangeness of strange attractors. *Physica D*, 9:189 – 208, 1983.

[11] P. Grassberger, T. Schreiber, and C. Schaffrath. Non-linear time sequence analysis. *International Journal of Bifurcation and Chaos*, 1:521 – 547, 1991.

[12] M. Hénon. A two-dimensional map with a strange attractor. *Commun. Math. Phys.*, 50:69 – 77, 1976.

[13] E. J. Kostelich and J. A. Yorke. Noise reduction: Finding the simplest dynamical system consistent with the data. *Physica D*, 41:183 – 196, 1990.

[14] D. Kugiumtzis. State space reconstruction parameters in the analysis of chaotic time series - the role of the time window length. *Physica D*, 95:13 – 28, 1996.

[15] D. Kugiumtzis. Correction of the correlation dimension for noisy time series. *International Journal of Bifurcation and Chaos*, in press, 1997.

[16] D. Kugiumtzis, B. Lillekjendlie, and N. Christophersen. Chaotic time series part I: Estimation of some invariant properties in state space. *Modeling, Identification and Control*, 15(4):205 – 224, 1994.

[17] D. P. Lathrop and E. J. Kostelich. *Physical Reviews A*, 40:431, 1989.

[18] B. Lillekjendlie, D. Kugiumtzis, and N. Christophersen. Chaotic time series part II: System identification and prediction. *Modeling, Identification and Control*, 15(4):225 – 243, 1994.

[19] E. N. Lorenz. Deterministic nonperiodic flow. *J. Atmos. Sci.*, 20:130, 1963.

[20] A. Medio. *Chaotic Dynamics: Theory and Applications to Economics*. Cambridge University Press, Cambridge, 1992.

[21] A. I. Mees, P. E. Rapp, and L. S. Jennings. Reply to "comment on 'singular-value decomposition and embedding dimension'". *Physical Review A*, 37(12):5006, 1988.

[22] E. Ott. *Chaos in Dynamical Systems*. Cambridge University Press, Cambridge, 1993.

[23] N. H. Packard, J. P. Crutchfield, J. D. Farmer, and R. S. Shaw. Geometry from a time series. *Physical Review Letters*, 45:712, 1980.

[24] M. I. Rabinovich and A. L. Fabrikant. Stochastic self-modulation of waves in nonequilibrium media. *Sov. Phys. JETP*, 50:311, 1979.

[25] P. L. Read. Applications of singular system analysis to 'baroclinic chaos'. *Physica D*, 58:455 – 468, 1992.

[26] O. E. Rössler. An equation for hyperchaos. *Physics Letters A*, 71(2 – 3):155 – 157, 1979.

[27] T. Sauer and J. A. Yorke. How many delay coordinates do you need? *International Journal of Bifurcation and Chaos*, 3(3):737 – 744, 1993.

[28] T. Sauer, J. A. Yorke, and M. Casdagli. Embedology. *Journal of Statistical Physics*, 65:579 – 616, 1991.

[29] A. S. Sharma, D. Vassiliadis, and K. Papadopoulos. Reconstruction of low-dimensional magnetospheric dynamics by singular spectrum analysis. *Geophysical Research Letters*, 20(5):335 – 338, 1993.

[30] F. Takens. Detecting strange attractors in turbulence. In D. A. Rand and L. S. Young, editors, *Dynamical Systems and Turbulence, Warwick 1980*, Lecture Notes in Mathematics 898, pages 366 – 381. Springer, Berlin, 1981.

[31] R. Vautard, P. Yiou, and M. Ghil. Singular-spectrum analysis: A toolkit for short, noisy chaotic signals. *Physica D*, 58:95 – 126, 1992.