An Approach to the Design of Sparse Array Systems

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Abstract — Sparse arrays have been proposed for two-dimensional arrays for three-dimensional ultrasound imaging in order to reduce the number of channels in the system. Such arrays have been designed by picking array elements in a random fashion, either according to a uniform or a Gaussian distribution. A random array can have large variations in the level of the maximum sidelobe.

A method for optimization of the sidelobe level of 1-D sparse arrays has been demonstrated. This shows that weighting can give responses that resemble filled Dolph-Chebyshev arrays. The initial thinning pattern is of less importance for the final result, but the less ideal the unweighted pattern is, the more dynamic range is required from the weight function.

Introduction

Three dimensional ultrasound imaging has been demonstrated to give unique clinical information. It is now considered to be an imaging mode with a definitive future. One of the striking clinical examples that demonstrate the mode's potential is three-dimensional cineloops of moving heart valves. This kind of imaging is presently done with arrays that are mechanically scanned in the second dimension. In the future two-dimensional electronically scanned and focused arrays will give advantages such as simpler use, easier access and no moving parts.

Two-dimensional arrays present several challenges in technology, the major one from a systems point-of-view being the large number of channels that needs to be handled. Typically one would desire $64 \times 64 = 4096$ channels. Thinning of the array elements to give a sparse array has therefore been proposed in order to reduce this number to 1/4 or even less [6]. Such arrays have been designed by picking array elements in a random fashion, either according to a uniform or a Gaussian distribution. A random array can have large variations in the level of the maximum sidelobe level, therefore a search has been proposed in order to select the configuration with lowest maximum sidelobe [9].

Sparse arrays have been researched for a long time for applications in radar, sonar, and geophysics. Such arrays have been used as a means of obtaining maximum resolution with a minimal number of elements, using the coarray as a means of quantifying performance [8]. Several researchers have also investigated ways to control the sidelobe levels for sparse arrays. Almost all of the optimization methods are based on the minimax criterion, i.e. equiripple behavior in the mainlobe and sidelobe regions. The optimization problem for a one-dimensional line array can be approached from several different angles:

1. Uniform element spacing and optimization of element weights. This leads to conventional Dolph-Chebyshev weighting.
2. Unity weights and optimization of element spacings. It is demonstrated in [4] that by using a nonlinear minimax optimization procedure, the element spacings can be regarded as continuous variables and convergence is achieved.
3. Fixed pattern of non-uniformly spaced elements and optimization of element weights [2, 3].
4. Simultaneous optimization of positions and weights under the constraint to minimize the number of elements for a symmetric array [5, 7]. The search for positions is simplified by assuming a fixed underlying grid.

Optimization Method

Due to the importance of the coarray in designing sparse array geometries our first approach was to base the design on the coarray. The coarray is defined as the correlation of the aperture of an array. It is also the inverse Fourier transform of the far-field beam pattern of the array. Usually the coarray has been used to design arrays with as high resolution as possible. This is equivalent to having a coarray which is as uniform as possible, and which spans the maximum number of lags. Thus the desired coarray is equal to the number of elements for lag zero, and unity for all other lags. This is not achievable in practise for arrays with more than four elements, but approximations to the ideal has given rise to the concepts of minimum redundancy arrays and nonredundant minimum missing-lag arrays [8].

The design criterion here is a bit different since it is important to control not only detail resolution, but also contrast resolution. Thus one must have control over the sidelobe levels. As demonstrated later, the coarray is not a good indicator of that. Instead one must use design methods that do the optimization directly on the spatial response.

The method described here is based on a generalization of the Remez exchange method to an irregular grid ([2, 3]). The method can be used for thinned arrays based on an equi-spaced underlying grid. The method can also be used for unequally spaced arrays such as elements spaced according to the abscissas of a Gaussian quadrature, or symmetric spacings that follow a geometric series. The only restrictions are that the weights are symmetric and that the number of sidelobe peaks to control are approximately equal to half the number of coefficients.

The objective is to minimize the weighted approximation error between a desired response and the array response. Due to the similarities with FIR filter design, the minimax or Chebyshev criterion can also be used for array design. For symmetric FIR filters an efficient algorithm is given [1]. However, in our application we will also optimize sparse and non-equally spaced arrays. We have therefore modified the Remez algorithm to be able to this. Our objective is thus the minimax criterion:

$$min\Big\{max_{u\in F}|W(u_k)\big(D(u_k)-P(u_k)\big)|\Big\} \quad = \quad \delta$$

where F is the set of frequencies where the response is optimized, W is the error weighting and D is the desired response. The polynomial form of the array pattern used in the approximation is:

$$P(u) = \sum_{k=0}^{r} \alpha_k \cos(\xi_k u) \qquad (1)$$

$$\text{where} \quad \xi_k = \frac{2x_k}{\lambda}, \quad u = \pi \sin\phi$$

x_k : the sensor locations

λ : the design wavelength

For odd length arrays the weights to be applied on the individual sensors, are found from $w_0 = \alpha_0, \quad w_n = w_{-n} = \alpha_n/2$ and in the even case $w_n = w_{-n} = \alpha_n/2$. The alternation theorem allow us to formulate the problem as a matrix system [1]:

$$\boldsymbol{A}\vec{\boldsymbol{\alpha}} = \boldsymbol{D} \qquad (2)$$

where

$$\boldsymbol{A} = \begin{bmatrix} \cos\xi_0 u_0 & \cdots & \cos\xi_r u_0 & \frac{\delta}{W(u_0)} \\ \cos\xi_0 u_1 & \cdots & \cos\xi_r u_1 & \frac{-\delta}{W(u_1)} \\ \vdots & \vdots & \vdots & \vdots \\ \cos\xi_0 u_{r+1} & \cdots & \cos\xi_r u_{r+1} & \frac{(-1)^{r+1}\delta}{W(u_{r+1})} \end{bmatrix}$$

$$\vec{\boldsymbol{\alpha}} = \begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_r & \delta \end{bmatrix}^T$$

$$\boldsymbol{D} = \begin{bmatrix} D(u_0) & D(u_1) & \ldots & D(u_r) & D(u_{r+1}) \end{bmatrix}^T$$

The coefficients that form the solution to this system can then be used in Eq. 1 to calculate either the response or the approximation error. Now a search must be performed to find the extremums of the response function or the error function. For sufficient narrow mainlobes this search can be simplified to search for extremums in the stopband of the functions only. This will give a saving in the number of operations. We have differentiated the response and used the sign changes (i.e. v_i for $i = 1, 2, \ldots, M$ where M is the number of sign changes) as candidates for new extremums. Now the $r+1$ extremum candidates with the largest error are chosen. The initial extremums u_p and u_s in the pass- and stopband respectively, are members of this set. The updated extremal frequencies are then used in the next iteration and convergence is normally reached after only a few iterations (e.g. $4-6$). For wider mainlobes and when steering is applied, the search has to be extended to check the passband extremums more closely, since there will be alternations there also.

Results

The initial array is a 65-element equi-spaced array with an aperture of 32.5λ. The element positions have been perturbed and in Fig. 1 the weighting and response of the array are shown.

The equi-spaced array has then been thinned by drawing elements with a Gaussian probability density function across the aperture i.e. the further away from the center, the larger probability to be thinned. The resulting element positions are evident from Fig. 2. This figure also shows weights and the resulting response after optimization. The optimization criterion was to have approximately the same sidelobe level as the previous case.

Figure 3 shows the improvement in sidelobe level obtained by optimal weighting. The sidelobes are however only controlled in a region approximately equal to the inverse sine of the ratio of $N_{thinned}$ and N_{filled}. Therefore there is a large grating-lobe like response in the region $50° - 90°$.

In the next figure (Fig. 4) the array has been thinned by a uniform propability distribution. Now the array has only 31 elements.

The same 65-element array has also been thinned to 31 elements by Gaussian thinning. The result of the thinning and the optimization of the weights is shown in Fig. 5.

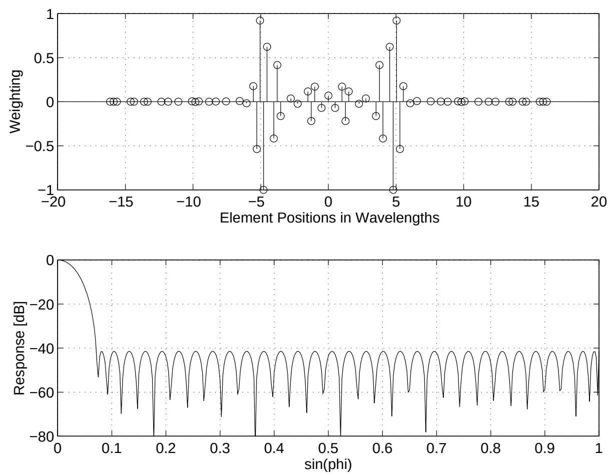

Figure 1: Characteristics of a perturbed 65-element equispaced array with aperture 32.5λ.

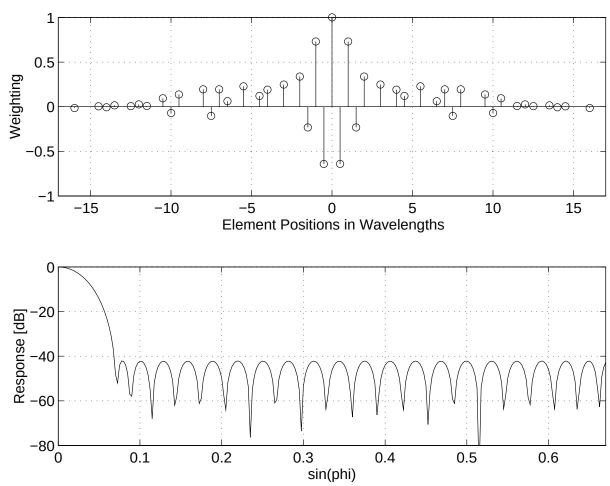

Figure 2: Weighting and response of a 65 element equispaced array with aperture 32.5λ thinned in a Gaussian way to 45 elements.

A comparison between the parameters of the various arrays is shown in the table to the right. The dynamic range is the ratio of the largest and the smallest weight value.

Finally a comparison between the coarrays of the weighted 65-element equi-spaced array and the weighted perturbed 65-element array is included in Fig. 6. It shows how different the coarrays can be even when the angular

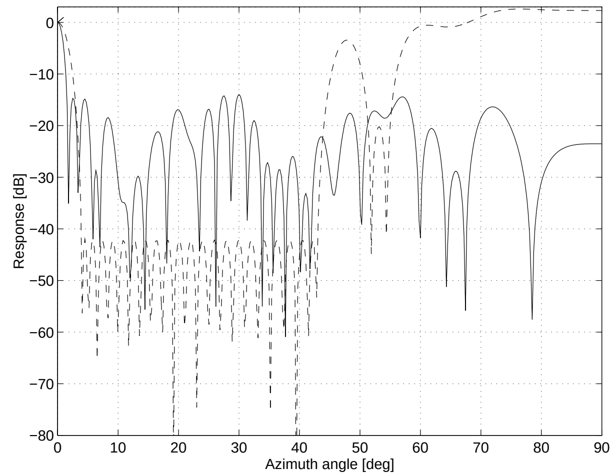

Figure 3: Comparison between unweighted (solid line) and weighted (dashed line) response for 65 element array thinned to 45 elements according to a Gaussian distribution.

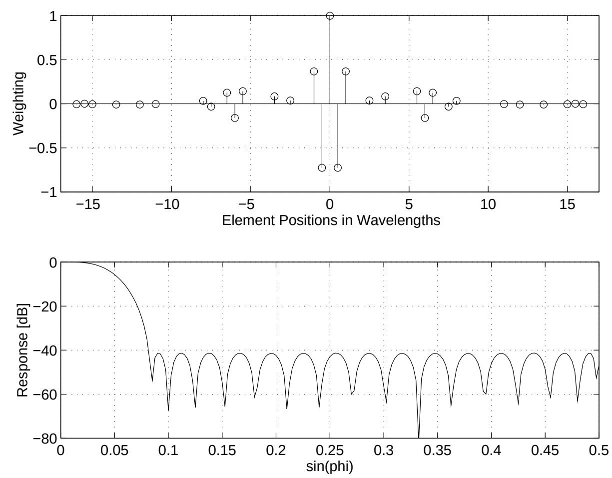

Figure 4: Weighting and response of a 65 element equispaced array with aperture 32.5λ uniformly thinned to 31 elements.

Type	Beamwidth $-6dB$	Dynamic Range	Sidelobe Level
65 Filled	3.0°	20.6 dB	−41.3 dB
65 Perturbed	4.2°	84.9 dB	−41.5 dB
45 Gauss	3.9°	47.5 dB	−42.2 dB
45 Uniform	4.3°	53.8 dB	−41.3 dB
31 Gauss	4.2°	52.0 dB	−40.8 dB
31 Uniform	5.8°	65.7 dB	−41.5 dB

responses are almost similar. This is caused by the differences in phase between the two responses, something which is not usually shown for an angular response.

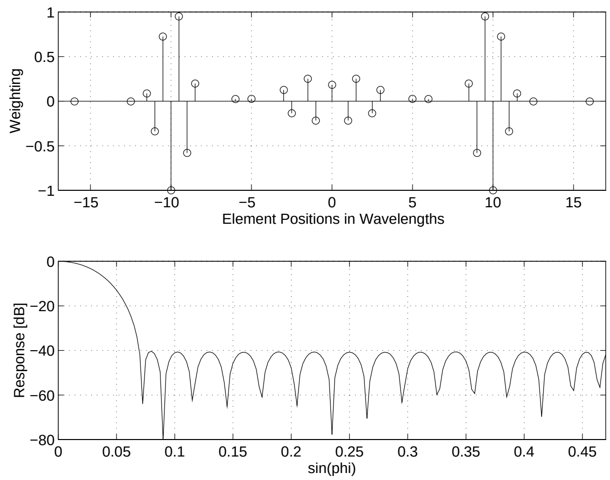

Figure 5: Weighting and response of a 65 element equispaced array with aperture 32.5λ thinned to 31 elements in a Gaussian manner.

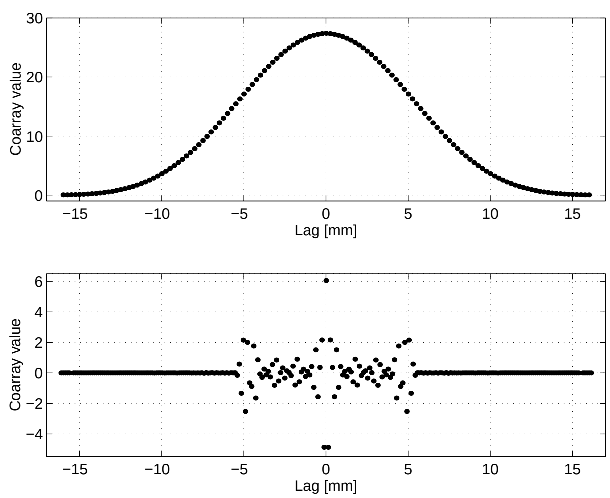

Figure 6: Comparison between difference coarrays of Dolph-Chebyshev (upper panel) and perturbed Dolph-Chebyshev array (lower panel).

Conclusion

A method for optimization of the sidelobe level of 1-D sparse arrays has been demonstrated. The method is based on the Remez exchange algorithm and is able to optimize the sidelobes for arrays with perturbed positions and for arrays with thinned elements. However the more elements that are removed, the more limited the angular region with controlled sidelobes will be.

The initial thinning or perturbation pattern is of less importance for the final result. However the poorer the unweighted response is, the more dynamic range is required from the weight pattern.

In order to control the sidelobes in the full angular region for 2-D arrays, an algorithm with less stringent requirements on the final result than the equiripple criterion is needed. This implies an algorithm based on constrained optimization.

References

[1] T. W. Parks and J. H. McClellan, "Chebyshev Approximation for Nonrecursive Digital Filters with Linear Phase", *IEEE Trans. Circuit Theory*, vol 19, no 2, pp 189-194, 1972.

[2] R. Streit, "Sufficient conditions for the existence of optimum beam patterns for unequally spaced linear arrays with an example," *IEEE Trans. Antennas Propagat.*, pp 112-115, Jan. 1975.

[3] R. Streit, "Optimized symmetric line arrays," *IEEE Trans. Antennas Propagat.*, pp 860-862, Nov. 1975.

[4] H. Schjær-Jacobsen and K. Madsen, "Synthesis of nonuniformly spaced arrays using a general nonlinear minimax optimization method," *IEEE Trans. Antennas Propagat.*, pp 501-506, July 1976.

[5] N. M. Mitrou, "Results on nonrecursive digital filters with nonequidistant taps," *IEEE Trans. Acoust., Speech, Sign. Proc.*, vol. ASSP-33, no 6, pp. 1621-1624, Dec. 1985.

[6] D. H. Turnbull and F. S. Foster, "Beam steering with pulsed two-dimensional transducer arrays," *IEEE Trans. Ultrason., Ferroelec., and Freq. Contr.*, vol 38, no 4, pp 320-333, July 1991.

[7] R. Leahy and B. D. Jeffs, "On the design of maximally sparse beamforming arrays," *IEEE Trans. Antennas Propagat.*, vol 39, no 8, pp 1178-1187, Aug. 1991.

[8] D. H. Johnson and D. E. Dudgeon, *Array signal processing: Concepts and techniques,* Prentice-Hall, 1993.

[9] R. E. Davidsen and S. W. Smith, "Sparse geometries for two-dimensional array transducers in volumetric imaging," in *Proc. IEEE 1993 Ultrason. Symp.*, pp 1091-1094, Nov. 1993.