Optimization of the Beampattern of 2D Sparse Arrays by Weighting

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***Abstract* — A method is presented which optimizes weights of general planar 1D and 2D symmetric full and sparse arrays.**

The objective is to find a weighting of the array elements which gives the minimum sidelobe level of the array pattern in a specified region - the stopband. The sidelobe level is controlled on a discrete set of points from this region. The method minimizes the Chebyshev norm of the sidelobe level. The method is based on linear programming and is solved with the simplex method.

The method removes the large fluctuation in sidelobe level which characterizes random sparse arrays. Examples of optimal weighted 1D and 2D planar arrays are presented.

I. Introduction

2D arrays in ultrasound represent a technological challenge not the least because of the high channel count [7]. For this reason sparse array methods, where elements are removed by thinning, are considered to be necessary [8]. However this will result in an often unacceptably high sidelobe level. Two different approaches to overcome this problem have been proposed. The first is optimization of the two-way beampattern without consideration for the individual transmit and receive beampatterns [6]. The second is to optimize the transmit and receive beampatterns independently.

In the first approach the transmit energy may often be spread over a large angular region. In medical applications the peak intensity is limited by safety concerns and thus it may not be possible to achieve a good enough signal to noise ratio. In addition the beam sharpening resulting from both a narrow transmit and receive beam is not achieved.

In the second approach, the one-way beampattern of the thinned array is optimized. This has been done by searching for the thinning pattern that minimizes the peak sidelobe level [3], or that is the best approximation to the full array's beampattern [5].

In [4] it was proposed instead to optimize the beampattern by finding the best element weights for a given thinning pattern. In this paper the results are extended from 1D to 2D arrays and improved results are reported since linear programming is used for the optimization. In this way a constrained optimization is performed where a specific beamwidth is achieved while minimizing the maximum sidelobe.

A. The beam pattern of a planar array

The beampattern of a $2N$ element array is given as

$$W(\vec{k}) = \sum_{n=1}^{2N} w_n e^{j\vec{k}\cdot\vec{x}_n} \tag{1}$$

where the array element locations are $\vec{x}_n \in \mathbb{R}^3$ with the corresponding weights $w_n \in \mathbb{R}$. The wavenumber vector $\vec{k} \in \mathbb{R}^3$ has amplitude $|\vec{k}| = 2\pi/\lambda$ where λ is the design wavelength.

The elements of a 2D planar array are located in the xy-plane with element n at $\vec{x}_n = (x_n, y_n, 0)$. Define also the unit direction vector with the following rectangular coordinates $\vec{s}_{\phi,\theta} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$.

To ensure a real beampattern and a real optimization problem, symmetric arrays with symmetric real weighting are considered [9]. The beampattern is:

$$\begin{aligned} W(\phi,\theta) &= 2\sum_{n=1}^{N} w_n \cos\left(\frac{2\pi}{\lambda}\vec{s}_{\phi,\theta}\cdot\vec{x}_n\right) \qquad (2) \\ &= 2\sum_{n=1}^{N} w_n \cos\left(\frac{2\pi}{\lambda}\sin\phi(x_n\cos\theta + y_n\sin\theta)\right) \end{aligned}$$

which gives the array response to a monochromatic wave from direction (ϕ,θ) in space (figure 1).

Using matrix notation one gets:

$$W(\phi,\theta) = \mathbf{v}(\phi,\theta)^T\mathbf{w} \tag{3}$$

where $\mathbf{w} = [w_1 \cdots w_N]^T$ are the element weights and the kernel vector $\mathbf{v}(\phi,\theta)$ is given as $\mathbf{v}(\phi,\theta) = \left[2\cos(\frac{2\pi}{\lambda}\vec{s}_{\phi,\theta}\cdot\vec{x}_1) \ \cdots \ 2\cos(\frac{2\pi}{\lambda}\vec{s}_{\phi,\theta}\cdot\vec{x}_N)\right]^T$.

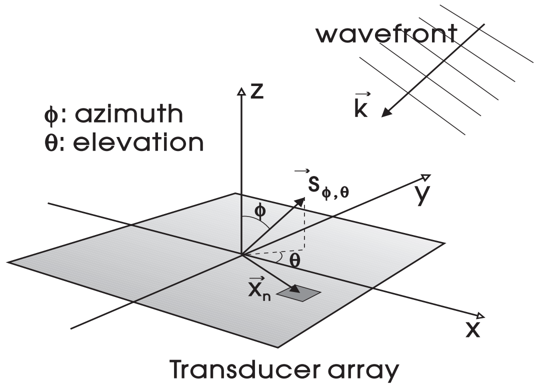

Figure 1: A 2D planar array with coordinate systems.

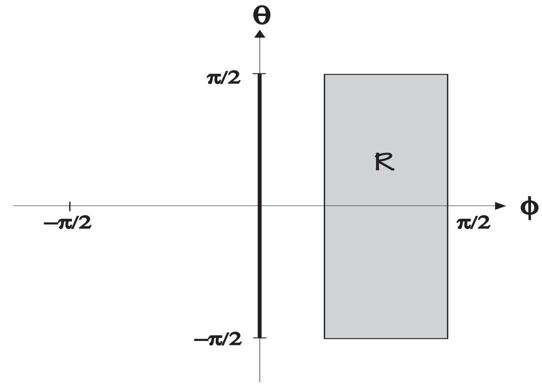

Figure 2: The optimization region $\mathcal{R}$ in the $\phi\theta$-plane.

II. Optimization problem

The objective is to minimize the sidelobe level in a continuous region $\mathcal{R}$ of the $\phi\theta$-plane (figure 2), which is the equivalent of a filter's stopband. Since the beampattern is symmetric about the θ-axis; $W(-\phi,\theta)=W(\phi,\theta)$, only the right halfplane is left for the optimization. The passband in this case is minimal in the sense that it consists only of the θ-axis.

A. Optimization formulation

The optimization problem may now be stated loosely as

$$\begin{array}{ll} \underset{\mathbf{w}}{\textit{Minimize}} & \text{array pattern level in stopband} \\ & \\ \textit{Subject to} & \text{normalized array pattern level} \\ & \text{in passband} \end{array} \tag{4}$$

The stopband region $\mathcal{R}$ is discretized into M gridpoints $R=\{(\phi_1,\theta_1)\ \cdots\ (\phi_M,\theta_M)\}$. The absolute array pattern level δ_s on the discrete set R is defined as

$$\delta_s = \max_{\phi,\theta\in R} |W(\phi,\theta)| \tag{5}$$

A more formal optimization formulation of (4) is now obtained with (5)

$$\begin{array}{ll} \underset{\mathbf{w}}{\textit{Minimize}}\ \delta_s & \\ & \\ \textit{Subject to} & W(0,\theta)=1 \\ & |W(\phi,\theta)| \le \delta_s \quad \forall\ (\phi,\theta)\in R \end{array} \tag{6}$$

This optimization problem can be transformed into a linear programming problem

$$\begin{array}{ll} \textit{Minimize} & \mathbf{c}^T\mathbf{x} \\ & \\ \textit{Subject to} & \mathbf{A}\mathbf{x} \le \mathbf{b} \end{array} \tag{7}$$

The problem in (6) may be written in standard form (7) by introducing block matrices for $\mathbf{c}, \mathbf{x}, \mathbf{A}$ and $\mathbf{b}$. Let the variable vector $\mathbf{x}$ consist of the weights $\mathbf{w}$ and the array pattern level indicator δ_s. The full linear program is stated as

$$\underset{\mathbf{w}}{\textit{Minimize}} \qquad \begin{bmatrix} \mathbf{0}_N^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \delta_s \end{bmatrix}$$

$$\textit{Subject to}$$

$$\begin{bmatrix} \mathbf{1}_N^T & 0 \\ \mathbf{-1}_N^T & 0 \\ \mathbf{v}(\phi,\theta)^T & -1 \\ -\mathbf{v}(\phi,\theta)^T & -1 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \delta_s \end{bmatrix} \le \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

$$\forall\ (\phi,\theta)\in R$$

where $\mathbf{0}_N^T$ and $\mathbf{1}_N^T$ are row vectors with N elements equal to 0 and 1 respectively.

B. Duality

Every linear program has another program associated with it. One of them is called the *primal* problem and the other the *dual* problem [1]. From linear programming theory, the duality theorem assures that if an optimal solution exists to either of them, then the other also has an optimal solution and the objective value coincides.

Since the solution to both programs are obtained by solving either one, it may be advantageous to solve the dual program rather than the primal itself.

The full linear program in (8) has an $\mathbf{A}$ matrix with $2M+2$ rows and $N+1$ columns. M is the number of discrete points on R and N are half the $2N$ element weights by symmetry. For most purposes $M>N$. With this kind of problem it is more effective to solve its dual [1]

$$
\begin{array}{lrcl}
\textit{Maximize} & \mathbf{b}^T\mathbf{y} & & \\
\textit{Subject to} & \mathbf{A}^T\mathbf{y} & = & \mathbf{c} \\
 & \mathbf{y} & \leq & \mathbf{0}
\end{array}
\tag{9}
$$

where $\mathbf{c}, \mathbf{x}, \mathbf{A}$ and $\mathbf{b}$ is as above. The optimal solution to the primal is established as a transform of the optimal dual solution. Let $\mathbf{y}^*$ be an optimal basic solution to the dual problem. Then an optimal solution $\mathbf{x}^*$ to the primal problem is

$$\mathbf{x}^* = \mathbf{A}_0^{-1}\mathbf{b} \tag{10}$$

$\mathbf{A}_0$ consists of the rows from $\mathbf{A}$ corresponding to the *basic* variables in $\mathbf{y}^*$.

III. Examples

A. 1D sparse array

A 3.5 MHz array with half lambda spacing, 64 elements and gaussian thinning to 48 elements was optimized. The beam patterns before and after optimization are shown in figure 3. Several such simulations were performed for various beamwidths and thinning patterns. The result is shown in figure 4. The curve starts leveling off at -17 to -18 dB sidelobe level. This is in the vicinity of the mean sidelobe level predicted for a random array given as the inverse of the number of elements [2] which is -16.8 dB for 48 elements.

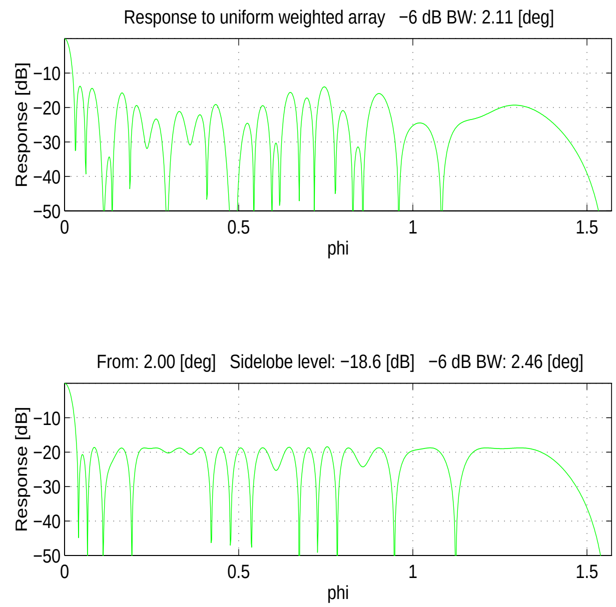

Figure 3: Beampattern before and after optimization for 64 element array thinned to 48 elements.

B. 2D sparse array

A 2D square array for 3.5 MHz with 16 by 16 elements with half wavelength spacing in both dimensions is considered. 50% thinning gives a beam pattern with a maximum sidelobe of -11 dB and a -6 dB beamwidth of 7-8 degrees (like the full array). Weighting results in a reduction of the sidelobe level to -17.7 dB with hardly any loss of beamwidth as shown in figure 5.

Finally a 64 by 64 element array with half lambda spacing at 3.5 MHz is considered. The array is inscribed in a circle giving a total of 3228 elements. The array is thinned to 404 elements (87.5 % thinning) and the response is optimized. The size of the problem is to find 202 weights plus the sidelobe level using 11520 control points for sidelobe control. The optimization took less than 5 minutes in CPLEX on a Silicon Graphics Power Challenge L. However to set up the problem in MATLAB took almost ten times the time needed for optimization. The result is shown in figure 6. The peak sidelobe is reduced from -9.5 dB to -17.4 dB without changing the beamwidth.

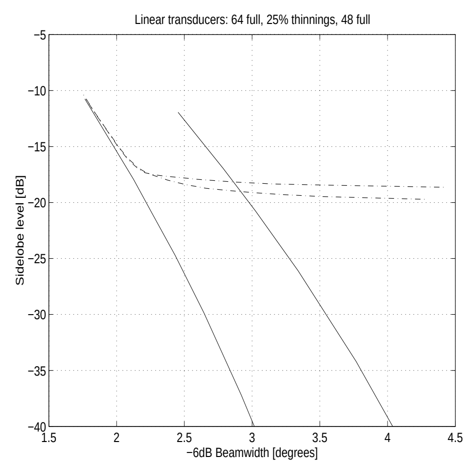

Figure 4: Sidelobe level as a function of beamwidth for 64 element and 48 element full arrays (solid lines) and for several realizations of 25% thinning of the 64 element array (dotted lines).

IV. Conclusion

The optimization method for finding weights is shown to work for 1D and 2D, and full and sparse arrays. The formulation is simple, and it is simple to include additional conditions like for instance a restriction to positive weights only.

Theory for random arrays predicts a mean sidelobe level given by the inverse of the number of elements. In practise however, the sidelobe level fluctuates much around this mean. With optimization of the weights it is possible to come quite close to the predicted value, especially for 1D arrays. Even for 2D sparse arrays a large reduction in peak sidelobe level is achieved.

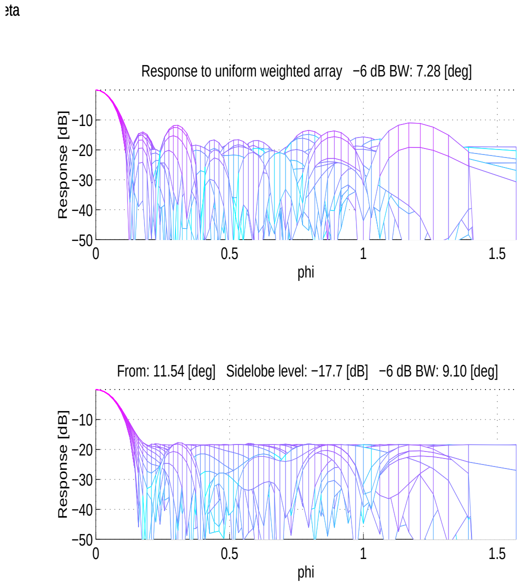

Figure 5: Beampattern for 16 by 16 element array thinned 50%. The unweighted (top) and the optimally weighted (bottom) responses are shown from the side.

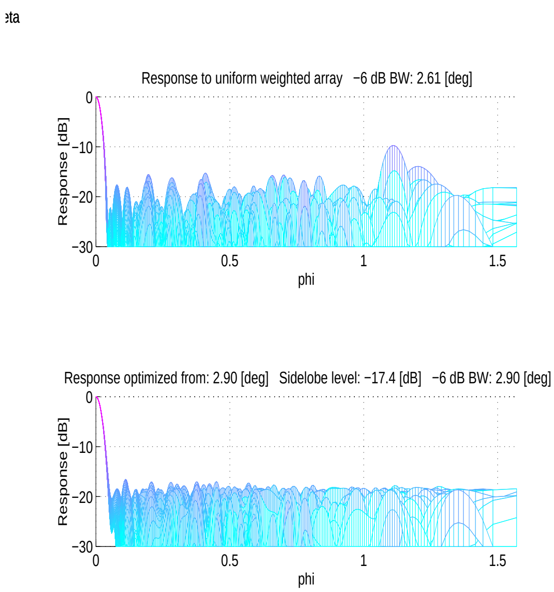

Figure 6: Beampattern for 64 by 64 element array first inscribed in a circle (3228 elements) and then thinned by 87.5% (404 active elements). The unweighted (top) and the optimally weighted (bottom) responses are shown from the side.

V. Acknowledgement

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References

[1] V. Chvátal, *"Linear programming"*, W. H. Freeman and Company, 1983.

[2] D. H. Johnson, D. E. Dudgeon *"Array signal processing"*, Prentice Hall, 1993.

[3] R. E. Davidsen, S. W. Smith, "Sparse geometries for two-dimensional array transducers in volumetric imaging", Proc. 1993 IEEE Symp. Ultrasonics, Baltimore, USA.

[4] J.O. Erstad, S. Holm, "An approach to the design of sparse array systems", Proc. 1994 IEEE Symp. Ultrasonics, Cannes, France.

[5] P. K. Weber, R. M. Schmidt. B. D. Tylkowski, J. Steck, "Optimization of random sparse 2-D transducer arrays for 3-D electronic beam steering and focusing", Proc. 1994 IEEE Symp. Ultrasonics, Cannes, France.

[6] G. R. Lockwood, F. S. Foster, "Optimizing sparse two-dimensional transdcuer arrays using an effective aperture approach", Proc. 1994 IEEE Symp. Ultrasonics, Cannes, France.

[7] B. A. J. Angelsen, H. Torp, S. Holm, K. Kristoffersen and T. A. Whittingham, "Which transducer array is best?", *Eur. Journ. Ultrasound,* vol 2, pp 151-164, 1995.

[8] S. Holm, "Medical ultrasound transducers and beamforming," Proc. Int. Congress on Acoustics, Trondheim, Norway, June 1995.

[9] B. Elgetun and S. Holm, "A method to optimize weighting of general planar arrays," Proc. Norwegian Signal Processing Conference, NORSIG-95, Sept. 1995.